A Call-by-name Calculus of Records and its Basic Properties

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1 Introduction

This technical report contains definitions and proofs of properties of a call-by-name calculus of records. The purpose of the calculus is to describe unordered collections of (possibly mutually recursive) named components. The system resembles a call-by-value calculus of records defined in [5], but the proof method used in [5] (based on the lift and project properties) as well as an extension of this method given in [6] fail for this calculus.

In this document we present the following:

1. the definition of the calculus - section 2.1.

2. the proof of confluence of evaluation relation in the calculus - section 3.

3. an attempt of a proof of the elementary lift/project diagram, as defined in [6] - section 4. The section defines and proves the elementary lift/project property (see 4.1). Unfortunately, the proof fails to find a non-trivial completion of the elementary lift/project diagram for the case of interaction of two substitutions in mutually recursive components: one in an evaluation context and the other one in a non-evaluation context. See case 3 of the two substitutions (SS) case of the proof of theorem 4.2.

   The completion of the diagram is trivial in the sense that it simply reverses both steps, which technically satisfies the property 4.1, but clearly does not allow for incorporating into an inductive proof (with the induction on the number of evaluation steps, as in the approach in [5]) or for proving termination properties, as in the approach in [6]. Example 4.7 shows why a non-trivial completion of the diagram is not possible.

This document does not discuss goals, motivation, and the framework. See other work by the authors for such a discussion ([5] contains the most comprehensive discussion).

2 Definitions

2.1 Call by Name Calculus of Records

The calculus is a two level system: the term level and the record level. Sets whose names start with T are at the term level, those that start with R are at the record level.

Definition 2.1. *The calculus of records is defined as follows:*

$$\begin{array}{lcl}
M \in TTerm & ::= & c \mid x \mid l \mid \bullet \mid \lambda x.M \mid M_1 \ @ \ M_2 \mid M_1 \ + \ M_2 \\
\mathbb{C} \in TContext & ::= & \square \mid \lambda x.\mathbb{C} \mid \mathbb{C} \ @ \ M \mid M \ @ \ \mathbb{C} \mid \mathbb{C} \ + \ M \mid M \ + \ \mathbb{C} \\
\mathbb{E} \in TEvalContext & ::= & \square \mid \mathbb{E} \ @ \ M \mid c \ + \ \mathbb{E} \mid \mathbb{E} \ + \ M \\
D \in RTerm & ::= & [l_1 \mapsto M_1, ..., l_n \mapsto M_n], l_i \neq l_j \ \textit{if} \ i \neq j \\
\mathbb{D} \in RContext & ::= & [l_1 \mapsto \mathbb{C}, l_2 \mapsto M_2, ..., l_n \mapsto M_n] \\
\mathbb{G} \in REvalContext & ::= & [l_1 \mapsto \mathbb{E}, l_2 \mapsto M_2, ..., l_n \mapsto M_n]
\end{array}$$

Here $M, N \in TTerm$ stands for terms, c are constants, x, y, z are variables (distinct from constants), l stands for labels (distinct from variables and constants), $\bullet$ is a black hole, $\lambda x.M$ is a lambda abstraction, $M_1 @ M_2$ is a function application, $M_1 + M_2$ is a binary operation on terms, $\square$ is a context hole, $\mathbb{C}$ is a general term context, $\mathbb{E}$ is a term evaluation context, $D \in Rterm$ is a record, $l \mapsto M$ is a binding (a component) in a record, where the term M is bound to the label l, $\mathbb{D}$ is a general record context, and $\mathbb{G}$ is a record evaluation context. We also use notation $\overline{\mathbb{C}}$ for a term non-evaluation context, i.e. by definition $\overline{\mathbb{C}} \in TContext \backslash TEvalContext$. Likewise $\overline{\mathbb{D}}$ is a record non-evaluation context defined as $\overline{\mathbb{D}} \in RContext \backslash REvalContext$.

Both levels of the calculus follow the call-by-name reduction strategy. We define a reduction relation $\rightarrow$ and evaluation relation $\Rightarrow$. On terms $\Rightarrow \subset \rightarrow \subseteq TTerm \times TTerm$, on records $\Rightarrow \subset \rightarrow \subseteq RTerm \times RTerm$. For both calculi $\hookrightarrow = \rightarrow \setminus \Rightarrow$. Note that we use the same notations for the relations at the term and at the record level.

Term Calculus Rules:

$$
\begin{array}{lll}
\lambda x.M @ N & \rightsquigarrow & M[x := N] \\
c_1 + c_2 & \rightsquigarrow & c_3 \text{ where } c_3 \text{ is the result of operation} \\
\mathbb{E}\{\bullet\} & \rightsquigarrow & \bullet \\
\mathbb{E}\{R\} & \Rightarrow & \mathbb{E}\{Q\} \text{ where } R \rightsquigarrow Q \\
\mathbb{C}\{R\} & \rightarrow & \mathbb{C}\{Q\} \text{ where } R \rightsquigarrow Q
\end{array}
$$

Definition 2.2. *The term included in the context on the left-hand side of the term calculus rules above is called the redex of the corresponding reduction. We use R as a metavariable for a redex.*

Intuitively, the redex is the subterm that gets reduced by the reduction. It is included in the context that remains unchanged by the reduction. Example: in the reduction $\lambda x.2 + \bullet \rightarrow \lambda x.\bullet$ the redex is $2 + \bullet$. In the evaluation step $1 + \lambda x.x @ 3 \Rightarrow 1 + 3$ the redex is $\lambda x.x @ 3$.

Record Calculus Rules:

We use $[l_i \overset{n}{\underset{i=1}{\mapsto}} M_i]$ as the abbreviation for $[l_1 \mapsto M_1, ..., l_n \mapsto M_n]$, $l \downarrow M$ to denote that in the record the term M is bound to the label l, i.e. the record contains a binding $l \mapsto M$.

$$
\begin{array}{llll}
\mathbb{G}\{R\} & \Rightarrow & \mathbb{G}\{Q\} \text{ where } R \rightsquigarrow Q & (TE) \\
\mathbb{G}\{l\} & \Rightarrow & \mathbb{G}\{N\} \text{ where } l \downarrow N,\ \mathbb{G} \neq [l \mapsto \mathbb{E}, \dots] & (SE) \\
\mathbb{D}\{R\} & \rightarrow & \mathbb{D}\{Q\} \text{ where } R \rightsquigarrow Q & (T) \\
\mathbb{D}\{l\} & \rightarrow & \mathbb{D}\{N\} \text{ where } l \downarrow N & (S) \\
[l_1 \mapsto \mathbb{E}\{l_1\}, ...] & \Rightarrow & [l_1 \mapsto \mathbb{E}\{\bullet\}, ...] & (B)
\end{array}
$$

Definition 2.3 (Notations for closures)**.**

1. *$\longrightarrow^*$, $\Longrightarrow^*$, $\hookrightarrow^*$ stand for reflexive transitive closures of the respective relations.*
2. *$\rightarrow^?$, $\Rightarrow^?$, and $\hookrightarrow^?$ stand for reflexive closures of the respective relations.*
3. *$\leftrightarrow$ stands for the reflexive symmetric transitive closure of $\rightarrow$.*

2.2 Non-confluence of $\rightarrow$

This calculus has the same example of non-confluence as its call-by-value version described in [5]. Originally this example was described in [2] in a somewhat different system. The record $[l_1 \mapsto \lambda x.l_2, l_2 \mapsto \lambda y.l_1]$ has two non-evaluation substitution redexes. By choosing each of the two redexes we obtain these two records: $[l_1 \mapsto \lambda x.\lambda y.l_1, l_2 \mapsto \lambda y.l_1]$ and $[l_1 \mapsto \lambda x.l_2, l_2 \mapsto \lambda y.\lambda x.l_2]$. No matter what substitutions we perform on the two records, they cannot reduce to a common one since in the first one both component will reference l_1, and in the second component they will both reference l_2.

Note that both reductions in this example are non-evaluation steps.

3 Confluence of $\Rightarrow$

Lemma 3.1. $\mathbb{C}_1\{\mathbb{C}_2\} = \mathbb{E}$ *if and only if both* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *are evaluation contexts.*

Proof. By induction on the structure of an evaluation context. □

Lemma 3.2. *If* $\mathbb{E}_1\{R_1\} = \mathbb{E}_2\{R_2\}$*, where* R_1, R_2 *are redexes, then either*

- $\mathbb{E}_1 = \mathbb{E}_2$ *and* $R_1 = R_2$ *or*
- $R_1 = \mathbb{E}'\{\bullet\}$*,* $R_2 = \mathbb{E}''\{\bullet\}$*, and* $R_1 = \mathbb{E}'''\{R_2\}$ *or* $R_2 = \mathbb{E}'''\{R_1\}$*.*

Proof. By the induction on the structure of a term using lemma 3.1. □

Lemma 3.2 is effectively saying that, with the exception of the black hole case, there may be at most one redex in an evaluation context in a term. Black hole redexes in an evaluation contexts may be nested within the same term.

Lemma 3.3. *If* $M = \mathbb{E}_1\{l_1\} = \mathbb{E}_2\{l_2\}$ *then* $\mathbb{E}_1 = \mathbb{E}_2$ *and* $l_1 = l_2$ *and* $M \neq \mathbb{E}\{R\}$ *for any* $\mathbb{E}$ *and* R*.*

Proof. By the induction on the structure of a term using lemma 3.1. □

Theorem 3.4 (One-step confluence of $\Rightarrow$)**.** *If* $D_1 \Rightarrow D_2$ *and* $D_1 \Rightarrow D_3$ *then there exists* D_4 *s.t.* $D_2 \Longrightarrow^* D_4$ *and* $D_3 \Longrightarrow^* D_4$*.*

This property is also known as *weak confluence.*

Proof. The proof is by cases on pairs of given evaluation steps. They are labeled by the rules, e.g. (TS) stands for the case when one of the steps is a term reduction (T) and the other one a substitution (S). Note that the cases are symmetric, i.e. (TS) is the same as (ST).

When considering cases, we often skip those where the two steps occur in two different components and do not depend on each other in any way since in these case the one-step confluence diagram can be trivially completed. For convenience we mark each step with the name of the rule that it follows. Even though these are evaluation steps, we use T and S instead of TE and SE for the rule names for simplicity.

- (TT): Case 1: the case when the two evaluations happen in different components of a record is trivial, and the confluence diagram can be trivially completed.

  Case 2: By lemma 3.2 the only case when the two evaluations happen in the same component is when the term reduction "destroys" an evaluation context around a black hole. In this case $\mathbb{E}_1\{\bullet\} = \mathbb{E}_2\{\mathbb{E}_3\{\bullet\}\}$, where $\mathbb{E}_1\{\bullet\}$ is the redex of one evaluation step, and $\mathbb{E}_3\{\bullet\}$ is the redex of the other (directly follows from lemma 3.2):

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{\mathbb{E}_1\{\bullet\}\}, \ldots] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\bullet\}, \ldots] & \end{array}$$

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{\mathbb{E}_2\{\mathbb{E}_3\{\bullet\}\}\}, \ldots] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\mathbb{E}_2\{\bullet\}\}, \ldots] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\bullet\}, \ldots] & \end{array}$$

- (TS):

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}_1\{M_1\}, l_2 \mapsto \mathbb{E}_2\{l_1\}] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}_1\{M_1\}, l_2 \mapsto \mathbb{E}_2\{\mathbb{E}_1\{M_1\}\}] & \stackrel{T}{\Rightarrow}\stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}_1\{M_1'\}, l_2 \mapsto \mathbb{E}_2\{\mathbb{E}_1\{M_1'\}\}] & \end{array}$$

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}_1\{M_1\}, l_2 \mapsto \mathbb{E}_2\{l_1\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}_1\{M_1'\}, l_2 \mapsto \mathbb{E}_2\{l_1\}] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}_1\{M_1'\}, l_2 \mapsto \mathbb{E}_2\{\mathbb{E}_1\{M_1'\}\}] & \end{array}$$

- (TB): The black hole can not be in the same component as the term reduction by lemma 3.3. that we are reducing. Trivially completion of confluence diagram similarly to (TT)

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{l_1\}, l_2 \mapsto M] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{l_1\}, l_2 \mapsto M'] & \stackrel{B}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\bullet\}, l_2 \mapsto M' & \end{array}$$

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{l_1\}, l_2 \mapsto M] & \stackrel{B}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\bullet\}, l_2 \mapsto M] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\bullet\}, l_2 \mapsto M'] & \end{array}$$

- (SS): By lemma 3.3 the two substitutions must happen in two different record components, i.e. the starting record has a form $]l_1 \mapsto \mathbb{E}_1\{l'\}, l_2 \mapsto \mathbb{E}_1\{l''\} \ldots]$. It must be the case that $l' \neq l_1$ and $l'' \neq l_2$, otherwise one the steps would be a black hole step. We have the following cases (where all l_i are distinct labels):

  1. $l' = l_3, l'' = l_4$,

2. $l' = l'' = l_3$,
3. $l' = l_2, l'' = l_1$.

Below are proofs for each of the three cases. Case 1: Trivial confluence diagram in this case:

$$[l_1 \mapsto \mathbb{E}_1\{l_3\}, l_2 \mapsto \mathbb{E}_2\{l_4\}, l_3 \mapsto M_1, l_4 \mapsto M_2]$$

Case 2: This one also trivially produces the confluence diagram:

$$[l_1 \mapsto \mathbb{E}\{l_3\}, l_2 \mapsto \mathbb{E}\{l_3\}, l_3 \mapsto M]$$

Case 3: If two labels depend on each other, both components become black holes:

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}_1\{l_2\}, l_2 \mapsto \mathbb{E}_2\{l_1\} \ldots] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}_1\{\mathbb{E}_2\{l_1\}\}, l_2 \mapsto \mathbb{E}_2\{l_1\} \ldots] & \stackrel{B}{\Rightarrow}\stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \bullet, l_2 \mapsto \mathbb{E}_2\{l_1\} \ldots] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \bullet, l_2 \mapsto \mathbb{E}_2\{\bullet\} \ldots] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \bullet, l_2 \mapsto \bullet \ldots] & \end{array}$$

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}_1\{l_2\}, l_2 \mapsto \mathbb{E}_2\{l_1\} \ldots] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}_1\{l_2\}, l_2 \mapsto \mathbb{E}_2\{\mathbb{E}_1\{l_2\}\} \ldots] & \stackrel{B}{\Rightarrow}\stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}_1\{l_2\}, l_2 \mapsto \bullet \ldots] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}_1\{\bullet\}, l_2 \mapsto \bullet \ldots] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \bullet, l_2 \mapsto \bullet \ldots] & \end{array}$$

The T cases here are the term "black hole" rule - the one that allows a black hole to consume an evaluation context around itself. We used lemma 3.1 to justify both B steps.

- (SB): One redex involves a substitution, the other a black hole.

  Case 1: Trivially Confluent.

$$[l_1 \mapsto \mathbb{E}_1\{l_1\}, l_2 \mapsto \mathbb{E}_2\{l_3\}, l_3 \mapsto M]$$

  Case 2:

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}_1\{l_1\}, l_2 \mapsto \mathbb{E}_2\{l_1\}] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}_1\{l_1\}, l_2 \mapsto \mathbb{E}_2\{\mathbb{E}_1\{l_1\}\}] & \stackrel{B}{\Rightarrow}\stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \bullet, l_2 \mapsto \mathbb{E}_1\{\mathbb{E}_1\{l_1\}\}] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \bullet, l_2 \mapsto \mathbb{E}_1\{\mathbb{E}_1\{\bullet\}\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \bullet, l_2 \mapsto \mathbb{E}_2\{\bullet\}] & \end{array}$$

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}_1\{l_1\}, l_2 \mapsto \mathbb{E}_2\{l_1\}] & \stackrel{B}{\Rightarrow}\stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \bullet, l_2 \mapsto \mathbb{E}_2\{l_1\}] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \bullet, l_2 \mapsto \mathbb{E}_2\{\bullet\}] & \end{array}$$

- (BB) The two evaluation steps must be in different components by lemma 3.3.

□ □

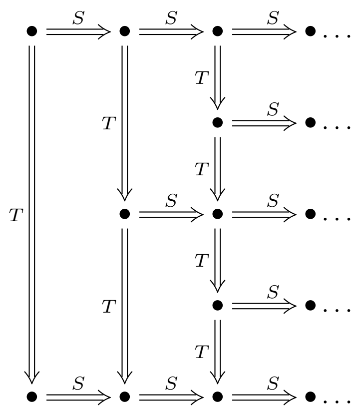
Weak confluence does not imply confluence because of a possibility of divergence of inductive diagrams. Below we show that divergence is not possible in our case.

Theorem 3.5. *In the calculus of records $\Rightarrow$ is confluent, i.e. given $D_1 \Longrightarrow^* D_2$ and $D_1 \Longrightarrow^* D_3$ there exists D_4 s.t. $D_2 \Longrightarrow^* D_4$ and $D_3 \Longrightarrow^* D_4$.*

Proof. In most cases of the proof of weak confluence given $D_1 \Rightarrow D_2$ and $D_1 \Rightarrow D_3$, there exists D_4 s.t. $D_2 \Rightarrow^? D_4$ and $D_3 \Rightarrow^? D_4$ (recall that $\Rightarrow^?$ stands for reflexive closure of $\Rightarrow$). This means that tiling the weak confluence diagrams leads to the proof of confluence without a possibility of divergence.

Below we consider the cases for which the property given in the previous paragraph does not hold and show that none of these cases cause divergence.

- (TS) In this case the T step may be duplicated by the substitution, but the S steps are not duplicated by the term reduction. The worst case scenario, therefore, looks like this:



  We observe that $\Rightarrow$ steps in this case satisfy the strip lemma (see [4]), and the strip lemma implies confluence of the relation.

- (SS) case 3. In this case the extra evaluation steps are added when two components depend on each other and both become black holes. The two extra steps are the black hole step and the term evaluation step that consumes the evaluation context around a black hole. Both steps can be repeated only a finite number of times since there is a limit on the number of black hole evaluation steps in a record: a record have no more black holes than the number of components, and once a black hole "consumes" a context around itself, there is no way to add something to the context that it's in. Therefore there is only a finite number of extra step related to black holes (B and the black hole TE steps) that a record may possibly generate, and thus such steps cannot cause divergence.

- (SB) case 2. Analogous to the previous case.

□ □

4 The Elementary Lift/Project Property

Definition 4.1. *A calculus has an elementary lift/project property if, given* $M_1 \Rightarrow M_2$ *and* $M_1 \hookrightarrow M_3$ *or* $M_3 \hookrightarrow M_1$*, there exists* M_4 *s.t.* $M_3 \Rightarrow M_4$ *and* $M_2 \leftrightarrow M_4$*.*

Theorem 4.2. *The call-by-name calculus of records has the elementary lift/project property.*

Lemma 4.3. *If* $M = \overline{\mathbb{C}}\{R\}$*, where* R *is a redex, it cannot be the case that* $R = \mathbb{C}\{R'\}$*,* R' *is a redex, and* $\overline{\mathbb{C}}\{\mathbb{C}\}$ *is an evaluation context.*

Proof. By induction on the structure of a term. □

Intuitively lemma 4.3 says that a non-evaluation redex cannot contain an evaluation redex.

Proof. The proof is by cases on the pairs of given reductions: an evaluation step and a non-evaluation step. By convention the rules are denoted so that the names of the non-evaluation step is first. For instance, (TS) denotes the case when the non-evaluation step is a term reduction, and the evaluation step is a substitution.

We use a 2-hole context notation to show a relative position of two non-nested subterms in a term. We use the same notation as the one hole context, $\mathbb{C}$ to denote a two-hole context. For instance, $\lambda x.x + 5$ can be seen as a two hole context $\lambda x.\square + \square$ filled with terms x and 5 (in this order). We also use multi-hole context notation, again denoting the context as $\mathbb{C}$. The definition is obvious.

For other notations see section 2.1.

By convention if a 2-hole context contains a redex in an evaluation context and a redex in another context, the evaluation redex/context pair is shown first. Note that by lemmas 3.2 and 3.3 there is at most one evaluation step from any component if not counting the black hole term redex (the case when a black hole "consumes" a context around itself).

- (TT) If the two reductions are in different components, the property clearly holds. Let us consider cases when the two reductions are in the same component. Recall that we have three cases of term reduction: application, operation, and the black hole. Note that an operation redex $c_1 + c_2$ cannot contain another redex.

  Below are all cases of non-trivial interactions of term reductions in the same component.

Case 1: non-overlapping redexes.

$$\begin{array}{ll}[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{R_1\}, \overline{\mathbb{C}}_2\{R_2\}\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{Q_1\}, \overline{\mathbb{C}}_2\{R_2\}\}] & \stackrel{T}{\rightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{Q_1\}, \overline{\mathbb{C}}_2\{Q_2\}\}] & \end{array}$$

$$\begin{array}{ll}[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{R_1\}, \overline{\mathbb{C}}_2\{R_2\}\}] & \stackrel{T}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{R_1\}, \overline{\mathbb{C}}_2\{Q_2\}\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{Q_1\}, \overline{\mathbb{C}}_2\{Q_2\}\}] & \end{array}$$

Case 2: the non-evaluation redex R is contained in the body of an application.

$$\begin{array}{ll}[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}\{x, ..., x, R\} \mathbin{@} M\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\mathbb{C}\{M, ..., M, R\}\}] & \stackrel{T}{\rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\mathbb{C}\{M, ..., M, Q\}\}] & \end{array}$$

$$\begin{array}{ll}[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}\{x, ..., x, R\} \mathbin{@} M\}] & \stackrel{T}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}\{x, ..., x, Q\} \mathbin{@} M\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\mathbb{C}\{M, ..., M, Q\}\}] & \end{array}$$

Case 3: the non-evaluation redex R is contained in the argument of the application.

$$\begin{array}{ll}[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x, ..., x\} \mathbin{@} \mathbb{C}_2\{R\}\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{R\}, ..., \mathbb{C}_2\{R\}\}\}] & \stackrel{T}{\longrightarrow}^* \\ [l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{Q\}, ..., \mathbb{C}_2\{Q\}\}\}] & \\ [l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x, ..., x\} \mathbin{@} \mathbb{C}_2\{R\}\}] & \stackrel{T}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x, ..., x\} \mathbin{@} \mathbb{C}_2\{Q\}\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{Q\}, ..., \mathbb{C}_2\{Q\}\}\}] & \end{array}$$

Case 4: The non-evaluation redex is contained in the evaluation context consumed by the black hole, i.e. the black hole redex $\mathbb{E}_1\{\bullet\} = \mathbb{C}_1\{\mathbb{E}_2\{\bullet\}, \mathbb{C}_2\{R\}\}$. The outer context must be an evaluation context by lemma 3.1.

$$\begin{array}{ll}[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{E}_2\{\bullet\}, \mathbb{C}_2\{R\}\}\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\bullet\}] & \end{array}$$

$$\begin{array}{ll}[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{E}_2\{\bullet\}, \mathbb{C}_2\{R\}\}\}] & \stackrel{T}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{E}_2\{\bullet\}, \mathbb{C}_2\{Q\}\}\}] & \stackrel{T}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\bullet\}] & \end{array}$$

- (TS) Term reduction is non-evaluation, substitution is evaluation. Again, we are skipping cases where the two reductions obviously don't interact.

Case 1:

$$\begin{array}{ll}[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\}, \mathbb{C}_2\{R\}\}, l_2 \mapsto M] & \overset{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{M\}, \mathbb{C}_2\{R\}\}, l_2 \mapsto M] & \overset{T}{\to} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{M\}, \mathbb{C}_2\{Q\}\}, l_2 \mapsto M] & \end{array}$$

$$\begin{array}{ll}[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\}, \mathbb{C}_2\{R\}\}, l_2 \mapsto M] & \overset{T}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\}, \mathbb{C}_2\{Q\}\}, l_2 \mapsto M] & \overset{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{M\}, \mathbb{C}_2\{Q\}\}, l_2 \mapsto M] & \end{array}$$

Case 2:

$$\begin{array}{lll}[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}_1\{R\}] & \overset{S}{\Rightarrow} & \\ [l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}_1\{R\}\}, l_2 \mapsto \overline{\mathbb{C}}_1\{R\}] & \overset{T}{\hookrightarrow} & \overset{T}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}_1\{Q\}\}, l_2 \mapsto \overline{\mathbb{C}}_1\{Q\}] & & \\ [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}_1\{R\}] & \overset{T}{\hookrightarrow} & \\ [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}_1\{Q\}] & \overset{S}{\Rightarrow} & \\ [l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}_1\{Q\}\}, l_2 \mapsto \overline{\mathbb{C}}_1\{Q\}] & & \end{array}$$

Note that the non-evaluation redex cannot contain the label that the substitution step substitutes into since the label must appear in an evaluation context. This is a consequence of lemma 3.1.

- (TB).

  Case 1: Trivial:

  $$[l_1 \mapsto \mathbb{E}_1\{l_1\}, l_2 \mapsto \overline{\mathbb{C}}\{R\}]$$

  Case 2:

  $$\begin{array}{ll}[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_1\}, \overline{\mathbb{C}}_2\{R_1\}\}] & \overset{B}{\Rightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{\bullet\}, \overline{\mathbb{C}}_2\{R_1\}\}] & \overset{T}{\hookrightarrow} \\ l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{\bullet\}, \overline{\mathbb{C}}_2\{Q_1\}\}] & \end{array}$$

  $$\begin{array}{ll}[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_1\}, \overline{\mathbb{C}}_2\{R_1\}\}] & \overset{T}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_1\}, \overline{\mathbb{C}}_2\{Q_1\}\}] & \overset{B}{\Rightarrow} \\ [l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{\bullet\}, \overline{\mathbb{C}}_2\{Q_1\}\}] & \end{array}$$

  No other cases are possible for the same reason as for (TS).

- (ST) We are skipping some trivial cases.

Case 1:

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{E}\{R\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{Q\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{Q\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{Q\}\}] &
\end{array}$$

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{E}\{R\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{R\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{R\}\}] & \stackrel{T}{\Rightarrow}\stackrel{T}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{Q\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{Q\}\}] &
\end{array}$$

Case 2:

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x, \ldots, x\} \mathbin{@} \mathbb{C}_2\{l_2\}\}, l_2 \mapsto M] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{l_2\}, \ldots, \mathbb{C}_2\{l_2\}\}\}, l_2 \mapsto M] & \stackrel{S}{\hookrightarrow} \ldots \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{M\}, \ldots, \mathbb{C}_2\{M\}\}\}, l_2 \mapsto M] &
\end{array}$$

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x, \ldots, x\} \mathbin{@} \mathbb{C}_2\{l_2\}\}, l_2 \mapsto M] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x, \ldots, x\} \mathbin{@} \mathbb{C}_2\{M\}\}, l_2 \mapsto M] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{M\}, \ldots, \mathbb{C}_2\{M\}\}\}, l_2 \mapsto M] &
\end{array}$$

Case 3:

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{R_1\}, \overline{\mathbb{C}}_2\{l_2\}\}, l_2 \mapsto M] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{Q_1\}, \overline{\mathbb{C}}_2\{l_2\}\}, l_2 \mapsto M] & \stackrel{S}{\rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{Q_1\}, \overline{\mathbb{C}}_2\{M\}\}, l_2 \mapsto M] &
\end{array}$$

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{R_1\}, \overline{\mathbb{C}}_2\{l_2\}\}, l_2 \mapsto M] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{R_1\}, \overline{\mathbb{C}}_2\{M\}\}, l_2 \mapsto M] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{Q_1\}, \overline{\mathbb{C}}_2\{M\}\}, l_2 \mapsto M] &
\end{array}$$

Case 4:

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{R_1\}, \overline{\mathbb{C}}_2\{l_1\}\}] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{Q_1\}, \overline{\mathbb{C}}_2\{l_1\}\}] & \stackrel{S}{\rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{Q_1\}, \overline{\mathbb{C}}_2\{\mathbb{C}_1\{\mathbb{E}_1\{Q_1\}, \overline{\mathbb{C}}_2\{l_1\}\}\}\}] & \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{R_1\}, \overline{\mathbb{C}}_2\{l_1\}\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{R_1\}, \overline{\mathbb{C}}_2\{\mathbb{C}_1\{\mathbb{E}_1\{R_1\}, \overline{\mathbb{C}}_2\{l_1\}\}\}\}] & \stackrel{T}{\Rightarrow}\stackrel{T}{\rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}_1\{Q_1\}, \overline{\mathbb{C}}_2\{\mathbb{C}_1\{\mathbb{E}_1\{Q_1\}, \overline{\mathbb{C}}_2\{l_1\}\}\}\}] &
\end{array}$$

Case 5: The label is nested within the evaluation redex

$$
\begin{array}{lr}
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{l_1, x, \ldots, x\} \mathbin{@} M\}] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{l_1, M, \ldots, M\}\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{E}\{\mathbb{C}_1\{l_1, M, \ldots, M\}\}, M, \ldots, M\}\}] &
\end{array}
$$

$$
\begin{array}{lr}
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{l_1, x, \ldots, x\} \mathbin{@} M\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{\mathbb{E}\{\lambda x.\mathbb{C}_1\{l_1, x, \ldots, x\} \mathbin{@} M\}, x, \ldots, x\} \mathbin{@} M\}] & \stackrel{T}{\Rightarrow}\stackrel{T}{\rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{E}\{\mathbb{C}_1\{l_1, M, \ldots, M\}\}, M, \ldots, M\}\} &
\end{array}
$$

Case 6: The label is nested within the evaluation redex.

$$
\begin{array}{lr}
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x, \ldots, x\} \mathbin{@} \mathbb{C}_2\{l_1\}\}] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{l_1\}, \ldots, \mathbb{C}_2\{l_1\}\}\}] &
\end{array}
$$

$$
\begin{array}{lr}
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x, \ldots, x\} \mathbin{@} \mathbb{C}_2\{l_1\}\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x, \ldots, x\} \mathbin{@} \mathbb{C}_2\{\mathbb{E}\{\lambda x.\mathbb{C}_1\{x, \ldots, x\} \mathbin{@} \mathbb{C}_2\{l_1\}\}\}\}] &
\end{array}
$$

Before we continue the proof for this case, we introduce a *multi-substitution* transformation which we abbreviate MS. This sequence of forward and backward term and substitution steps on a single component allows us to substitute the value of the entire component into several labels at once.

Definition 4.4. *Let $\mathbb{C}\{l, \ldots, l\}$ be a multi-hole context with all holes filled by a term l. $D_1 = [l \mapsto \mathbb{C}\{l, \ldots, l\}, \ldots] \stackrel{MS}{\rightarrow} [l \mapsto \mathbb{C}\{\mathbb{C}\{l, \ldots, l\}, \ldots, \mathbb{C}\{l, \ldots, l\}\}, \ldots] = D_2$ is called a multi-substitution step. Note that $\mathbb{C}$ does not have to contain all occurrences of l in the component, there may be other occurrences not captured by $\mathbb{C}$.*

Lemma 4.5. *$\stackrel{MS}{\rightarrow}$ can be represented as a sequence of forward and backward term and substitution steps on the component.*

Proof.

$$
\begin{array}{lr}
[l \mapsto \mathbb{C}\{l, \ldots, l\}, \ldots] & \stackrel{T}{\Leftarrow} \\
[l \mapsto (\lambda x.\mathbb{C}\{x, \ldots, x\}) \mathbin{@} l, \ldots] & \stackrel{S}{\hookrightarrow} \\
[l \mapsto (\lambda x.\mathbb{C}\{x, \ldots, x\}) \mathbin{@} ((\lambda x.\mathbb{C}\{x, \ldots, x\}) \mathbin{@} l), \ldots] & \stackrel{T}{\hookrightarrow} \\
[l \mapsto (\lambda x.\mathbb{C}\{x, \ldots, x\}) \mathbin{@} \mathbb{C}\{l, \ldots, l\}, \ldots] & \stackrel{T}{\Rightarrow} \\
[l \mapsto \mathbb{C}\{\mathbb{C}\{l, \ldots, l\}, \ldots \mathbb{C}\{l, \ldots, l\}\}, \ldots] &
\end{array}
$$

In the first step the name x is chosen in such a way that no free variables of the term are captured. □ □

Intuitively, the multi-substitution replaces multiple copies of a self-reference in a record component by a single reference. It is hard (if not impossible) to keep multiple copies of a self-referring labels "synchronized" if they are replaced one-by-one. However, reducing them to a single copy by a backward term reduction step and them substituting allows us to keep all of these labels "in synch". Example 4.6 with a single copy of a self-reference served as a motivation for the multi-substitution step.

Now we continue case 6 using the multi-substitution:

$$
\begin{array}{lr}
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x,\ldots,x\} \mathbin{@} \mathbb{C}_2\{l_1\}\}] & \overset{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{l_1\},\ldots,\mathbb{C}_2\{l_1\}\}\}] & \overset{MS}{\rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{\mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{l_1\},\ldots,\mathbb{C}_2\{l_1\}\}\}\},\ldots,\mathbb{C}_2\{\mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{l_1\},\ldots,\mathbb{C}_2\{l_1\}\}\}\}\}\}] &
\end{array}
$$

$$
\begin{array}{lr}
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x,\ldots,x\} \mathbin{@} \mathbb{C}_2\{l_1\}\}] & \overset{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x,\ldots,x\} \mathbin{@} \mathbb{C}_2\{\mathbb{E}\{\lambda x.\mathbb{C}_1\{x,\ldots,x\} \mathbin{@} \mathbb{C}_2\{l_1\}\}\}\}] & \overset{T}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\lambda x.\mathbb{C}_1\{x,\ldots,x\} \mathbin{@} \mathbb{C}_2\{\mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{l_1\},\ldots,\mathbb{C}_2\{l_1\}\}\}\}\}] & \overset{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{\mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{l_1\},\ldots,\mathbb{C}_2\{l_1\}\}\}\},\ldots,\mathbb{C}_2\{\mathbb{E}\{\mathbb{C}_1\{\mathbb{C}_2\{l_1\},\ldots,\mathbb{C}_2\{l_1\}\}\}\}\}\}] &
\end{array}
$$

Case 7: The cases of interactions between a term black hole reduction and a substitution step (omitting the case when the two steps appear in unrelated components) are listed below. The diagrams can be trivially completed.

$$
\begin{array}{l}
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{\bullet\},\mathbb{C}_2\{l_1\}\}] \\
[l_1 \mapsto \mathbb{E}\{\bullet\}, l_2 \mapsto \mathbb{C}\{l_1\}]
\end{array}
$$

- (SS) There are many different mutual positions of the two labels in this case. We list all of them and show proofs for the non-trivial ones. Recall that a label depending on itself in an evaluation context is a black hole reduction, not a substitution.

  Unfortunately, when two labels depend on each other, one in an evaluation context, and the other in a non-evaluation one (i.e. the record is $[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}]$) the system does not have a non-trivial completion of the diagram. The only way to complete the diagram is to reverse the original reductions. Therefore, the system effectively fails the desired property.

  Cases include:

$[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto M_1, l_3 \mapsto \overline{\mathbb{C}}\{l_4\}, l_4 \mapsto M_2]$	Trivial
$[l_1 \mapsto \mathbb{E}\{l_3\}, l_2 \mapsto \overline{\mathbb{C}}\{l_3\}, l_3 \mapsto M]$	Trivial
$[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_3\}, l_3 \mapsto M]$	Case 1 below
$[l_1 \mapsto \mathbb{E}\{l_3\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}, l_3 \mapsto M]$	Case 2 below
$[l_1 \mapsto \mathbb{E}\{l_3\}, l_2 \mapsto \overline{\mathbb{C}}\{l_2\}, l_3 \mapsto M]$	Trivial
$[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}]$	Case 3 below - no non-trivial completion
$[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_2\}]$	Case 4 below
$[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\},\mathbb{C}_2\{l_3\}\}, l_2 \mapsto M_1, l_3 \mapsto M_2]$	Trivial
$[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\},\mathbb{C}_2\{l_2\}\}, l_2 \mapsto M]$	Case 5 below
$[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\},\mathbb{C}_2\{l_1\}\}, l_2 \mapsto M]$	Case 6 below

There are no cases when l_1 depends on itself in an evaluation context (such as $[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_1\}, \mathbb{C}_2\{l_1\}\}]$) since these cases are black hole cases, not substitutions.

Case 1:

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_3\}, l_3 \mapsto M] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}\{l_3\}\}, l_2 \mapsto \overline{\mathbb{C}}\{l_3\}, l_3 \mapsto M] & \stackrel{S}{\hookrightarrow}\stackrel{S}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}\{M\}\}, l_2 \mapsto \overline{\mathbb{C}}\{M\}, l_3 \mapsto M] & \end{array}$$

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_3\}, l_3 \mapsto M] & \stackrel{S}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{M\}, l_3 \mapsto M] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}\{M\}\}, l_2 \mapsto \overline{\mathbb{C}}\{M\}, l_3 \mapsto M] & \end{array}$$

Case 2:

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{l_3\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}, l_3 \mapsto M] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{M\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}, l_3 \mapsto M] & \stackrel{S}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{M\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{M\}\}, l_3 \mapsto M] & \end{array}$$

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{l_3\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}, l_3 \mapsto M] & \stackrel{S}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{l_3\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{l_3\}\}, l_3 \mapsto M] & \stackrel{S}{\Rightarrow}\stackrel{S}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{M\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{M\}\}, l_3 \mapsto M] & \end{array}$$

Case 3:

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{S}{\Rightarrow} \\ [l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}\{l_1\}\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{S}{\Leftarrow} \\ [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{S}{\hookleftarrow} \\ [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{l_2\}\}] & \end{array}$$

$$\begin{array}{ll} [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{S}{\hookrightarrow} \\ [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{l_2\}\}] & \stackrel{S}{\Leftarrow} \\ [l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \end{array}$$

Unfortunately, in general this is a trivial completion of the diagram (see example 4.7 for the case that proves it). While it formally satisfies the elementary lift/project property, it does not allow us to make progress on the inductive proof.

Case 4:

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_2\}] & \stackrel{S}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}\{l_2\}\}, l_2 \mapsto \overline{\mathbb{C}}\{l_2\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}\{\overline{\mathbb{C}}\{l_2\}\}\}, l_2 \mapsto \overline{\mathbb{C}}\{l_2\} & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}\{\overline{\mathbb{C}}\{l_2\}\}\}, l_2 \mapsto \overline{\mathbb{C}}\{\overline{\mathbb{C}}\{l_2\}\} &
\end{array}$$

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_2\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{\overline{\mathbb{C}}\{l_2\}\}] & \stackrel{S}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\overline{\mathbb{C}}\{\overline{\mathbb{C}}\{l_2\}\}\}, l_2 \mapsto \overline{\mathbb{C}}\{\overline{\mathbb{C}}\{l_2\}\} &
\end{array}$$

Case 5:

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\}, \overline{\mathbb{C}}_2\{l_2\}\}, l_2 \mapsto M] & \stackrel{S}{\Rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{M\}, \overline{\mathbb{C}}_2\{l_2\}\}, l_2 \mapsto M] & \stackrel{S}{\rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{M\}, \overline{\mathbb{C}}_2\{M\}\}, l_2 \mapsto M] &
\end{array}$$

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\}, \overline{\mathbb{C}}_2\{l_2\}\}, l_2 \mapsto M] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\}, \overline{\mathbb{C}}_2\{M\}\}, l_2 \mapsto M] & \stackrel{S}{\Rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{M\}, \overline{\mathbb{C}}_2\{M\}\}, l_2 \mapsto M] &
\end{array}$$

Case 6:

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\}, \mathbb{C}_2\{l_1\}\}, l_2 \mapsto M] & \stackrel{S}{\Rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{M\}, \mathbb{C}_2\{l_1\}\}, l_2 \mapsto M] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{M\}, \mathbb{C}_2\{\mathbb{C}_1\{\mathbb{E}\{M\}, \mathbb{C}_2\{l_1\}\}\}\}, l_2 \mapsto M] &
\end{array}$$

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\}, \mathbb{C}_2\{l_1\}\}, l_2 \mapsto M] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{l_2\}, \mathbb{C}_2\{\mathbb{C}_1\{\mathbb{E}\{l_2\}, \mathbb{C}_2\{l_1\}\}\}\}, l_2 \mapsto M] & \stackrel{S}{\Rightarrow}\stackrel{S}{\rightarrow} \\
[l_1 \mapsto \mathbb{C}_1\{\mathbb{E}\{M\}, \mathbb{C}_2\{\mathbb{C}_1\{\mathbb{E}\{M\}, \mathbb{C}_2\{l_1\}\}\}\}, l_2 \mapsto M] &
\end{array}$$

- (SB)

  Case1:

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{E}\{l_1\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{B}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\bullet\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{\bullet\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{\bullet\}\}] &
\end{array}$$

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{E}\{l_1\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{E}\{l_1\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{l_1\}\}] & \stackrel{B}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\bullet\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{l_1\}\}] & \stackrel{S}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\bullet\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{\mathbb{E}\{\bullet\}\}\}] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \mathbb{E}\{\bullet\}, l_2 \mapsto \overline{\mathbb{C}}\{\mathbb{E}\{\bullet\}\}] &
\end{array}$$

Case 2:

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{C}\{\mathbb{E}\{l_1\}, \mathbb{C}_2\{l_1\}\}] & \stackrel{B}{\Rightarrow} \\
[l_1 \mapsto \mathbb{C}\{\mathbb{E}\{\bullet\}, \mathbb{C}_2\{l_1\}\}] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \bullet] &
\end{array}$$

$$\begin{array}{ll}
[l_1 \mapsto \mathbb{C}\{\mathbb{E}\{l_1\}, \mathbb{C}_2\{l_1\}\}] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \mathbb{C}\{\mathbb{E}\{l_1\}, \mathbb{C}_2\{\mathbb{C}\{\mathbb{E}\{l_1\}, \mathbb{C}_2\{l_1\}\}\}\}] & \stackrel{B}{\Rightarrow} \\
[l_1 \mapsto \mathbb{C}\{\mathbb{E}\{\bullet\}, \mathbb{C}_2\{\mathbb{C}\{\mathbb{E}\{l_1\}, \mathbb{C}_2\{l_1\}\}\}\}] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \bullet] &
\end{array}$$

Here $\bullet$ appears in an evaluation context, and therefore the whole component becomes a black hole.

□ □

Example 4.6. *Example of the ST case from the above proof. There is only one copy of l_1 in the term bound to l_1. This makes it easier to "synchronize" the term after the two different reductions.*

$$\begin{array}{ll}
[l_1 \mapsto (\lambda x.\lambda y.l_1) \mathbin{@} 2] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \lambda y.l_1] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto \lambda y.\lambda y.l_1] & \\
[l_1 \mapsto (\lambda x.\lambda y.l_1) \mathbin{@} 2] & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto (\lambda x.\lambda y.(\lambda x.\lambda y.l_1) \mathbin{@} 2) \mathbin{@} 2] & \stackrel{T}{\Rightarrow} \\
[l_1 \mapsto \lambda y.((\lambda x.\lambda y.l_1) \mathbin{@} 2)] & \stackrel{T}{\hookrightarrow} \\
[l_1 \mapsto \lambda y.\lambda y.l_1] &
\end{array}$$

Example 4.7 (Mutually Recursive Substitutions Fail Lift/Project)**.** *The following example illustrates that there is no non-trivial completion of the elementary lift/project diagram for two mutually recursive components, where one step is an evaluation step, and the other one is a non-evaluation step $[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \overline{\mathbb{C}}\{l_1\}]$:*

$$\begin{array}{ll}
[l_1 \mapsto l_2 + 1, l_2 \mapsto \lambda x.l_1 & \stackrel{S}{\Rightarrow} \\
[l_1 \mapsto (\lambda x.l_1) + 1, l_2 \mapsto \lambda x.l_1 &
\end{array}$$

$$\begin{array}{ll}
[l_1 \mapsto l_2 + 1, l_2 \mapsto \lambda x.l_1 & \stackrel{S}{\hookrightarrow} \\
[l_1 \mapsto l_2 + 1, l_2 \mapsto \lambda x.l_2 + 1 &
\end{array}$$

Since in the result of the evaluation step both components reference l_1 and in the result of a non-evaluation step both components reference l_2, no substitution can bring the two records together. Therefore it is unlikely that there is a non-trivial[1] *completion of the elementary lift/project diagram, and a trivial*

[1] i.e. the one that does not reverse both steps

completion cannot be incorporated into an inductive proof since no progress on evaluation has been made.

Note that the example is different from the case of two evaluation substitution steps $[l_1 \mapsto \mathbb{E}\{l_2\}, l_2 \mapsto \mathbb{E}\{l_1\}]$ where both components become black holes. It is also different from the non-confluence example in section 2.2 since in that example both steps are non-evaluation steps which does not affect the proof of the elementary lift/project property.

5 Conclusions

It turns out that in the call-by-name calculus of records the non-confluence extends to the case when one reduction is an evaluation step, and the other one is a non-evaluation step. This prevents us from using even the extended lift/project proof method (as described in [6]) for proving meaning-preservation of reduction-based transformation in this calculus. Note that failure to apply a specific proof method does not imply that reduction-based transformations are not meaning preserving. In fact, there is a reason to believe that the transformations may be meaning-preserving up to information contents, as in [1].

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