1 What Kinds of Numbers Are There?

1. Understanding numbers in terms of counting is not too difficult to understand, since we can count distinct objects ("I drank 5 coffees today and am feeling kinda twitchy!"):

   $1, 2, 3, 4, 5, 6, 7, 8, \ldots$ $\boxed{\text{natural numbers}}$

   I am using these kinds of numbers as a way of listing the points I am making.

2. Zero is actually quite tricky, since it is the absence of an object ("I have no homework due today!"). But we can decide zero is a number, which extends numbers to:

   $0, 1, 2, 3, 4, 5, 6, 7, \ldots$ $\boxed{\text{whole numbers}}$

   A powerful use of zero is as a placeholder to represent large numbers, like 1,000,000,000,000.

3. If we think of difference between numbers, ("The temperature change from last weekend to this weekend is −30°F. Brrrrrr!!") we can extend to negative numbers, since $40°\text{F} - 70°\text{F} = -30°\text{F}$. Now we have:

   $\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$ $\boxed{\text{integer numbers}}$

4. We can think of fractions of objects ("I ate $\frac{5}{6}$ of a pizza and now I feel sick."), and so can have new numbers like $\frac{5}{6}$, or more generally $\frac{p}{q}$ where p and q come from our earlier sets of numbers, the integers.

   $$\ldots, -1, \ldots -\frac{5}{6}, \ldots -\frac{2}{6}, \ldots 0, \ldots \frac{2}{6}, \ldots \frac{5}{6}, \ldots 1, \ldots \quad \boxed{\text{rational numbers}}$$

   The **numerator** is the number on top, and the **denominator** is the number on the bottom.

5. Another kind of number is an **irrational** number, which is a number that is not rational (so cannot be written as a ratio of integers).

   Examples of irrational numbers are $\sqrt{2} \sim 1.41421\ldots$, $e \sim 2.71828\ldots$, $\pi \sim 3.14159265359\ldots$, and less familiar ones like Euler's constant $\gamma \sim 0.577215664901532\ldots$. Irrational numbers, when written as decimals, do not terminate and do not repeat.

2 Fractions & Decimals

Mathematics: Fractions are preferred. For applied problems convert to decimal as the last step in your solution.

Science Courses: Fractions for theoretical problems. For applied problems decimals with error analysis.

In mathematics it is far preferable to work with fractions rather than decimals.

You must be very comfortable using fractions, since the manipulations used to work with fractions are the same manipulations used for more complicated algebraic expressions.

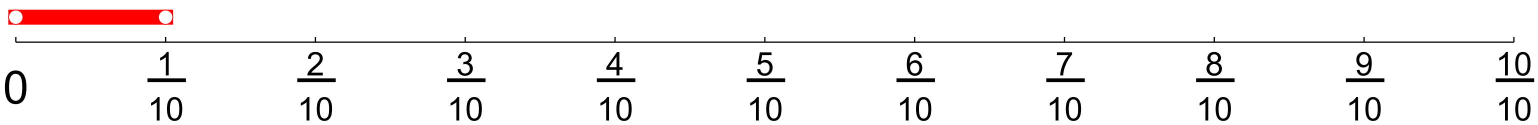
We work with the numbers 0,1,2,3,4,5,6,7,8,9 since we are working in base 10:

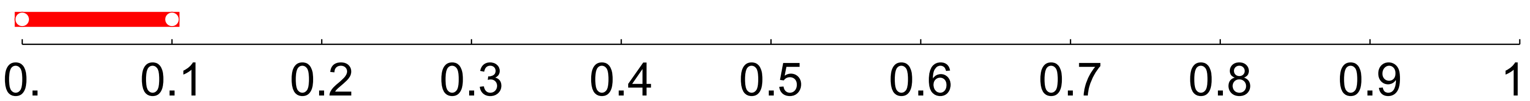
$$\mathbf{728} = \mathbf{7} \times 10^2 + \mathbf{2} \times 10^1 + \mathbf{8} \times 10^0 = 700 + 20 + 8 = 728$$

2.1 Relation Between Fractions and Decimals

The fraction 1/10 as a decimal

To begin, let's convert the fraction $\frac{1}{10}$ to a decimal. We can see that if we have ten lengths of $\frac{1}{10}$ we would have a total length of 1 on a ruler:





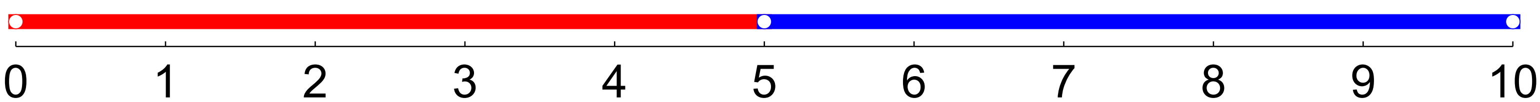
Therefore $\dfrac{1}{10} = 0.1$.

The fraction 1/5 as a decimal

To understand how other decimals and fractions are related, let's think of a ruler than has a total length of 10 units (it doesn't matter what the units are, but you could think of the units as inches or centimeters).

We choose a ruler with 10 units since we commonly work in base 10.

If we wanted to understand what the fraction $\frac{1}{5}$ means, we need to think about how many pieces of length 5 would fit along our ruler. We see that the answer is, of course, 2 pieces of length 5 fit on the ruler.

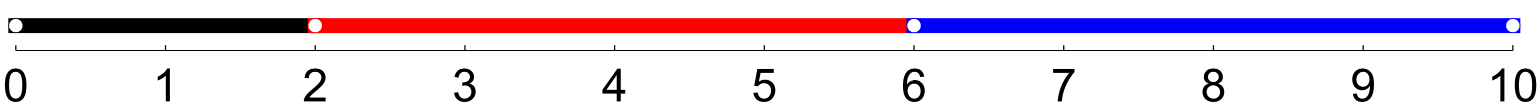


This means

$$\frac{1}{5} = 2 \times \frac{1}{10} = 2 \times (0.1) = 0.2.$$

The fraction 1/4 as a decimal

If we wanted to understand what the fraction $\frac{1}{4}$ means, we need to think about how many pieces of length 4 would fit along our ruler. We see that the answer is, of course, 2 and a half pieces of length 4 fit on the ruler.

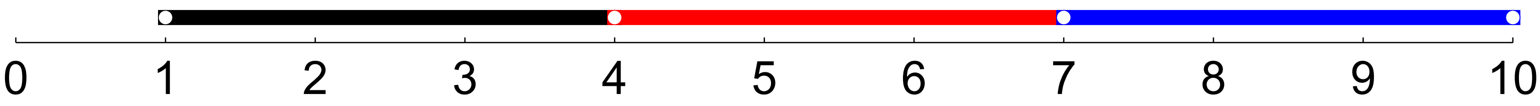


This means

$$\frac{1}{4} = 2.5 \times \frac{1}{10} = 0.25.$$

The fraction 1/3 as a decimal

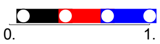
Here's a trickier one. If we wanted to understand what the fraction $\frac{1}{3}$ means, we need to think about how many pieces of length 3 would fit along our ruler. We see that we can fit three pieces, but then have some space left over.



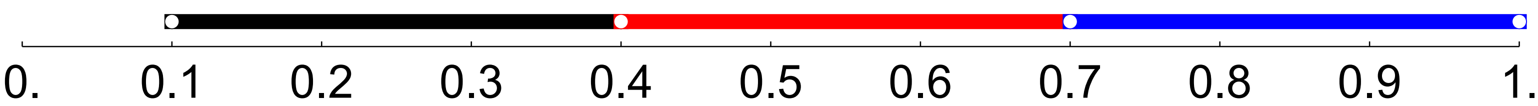
This means we can only say that approximately

$$\frac{1}{3} \sim 3 \times \frac{1}{10} = 0.3.$$

What we can do to get a better approximation is divide up the little piece of length one we had left over.



That's hard to see! I am going to zoom in to make it bigger, but notice the scale on the bottom now goes from 0 to 1, not 0 to 10. You can think of this as a new ruler, but with a different length. Notice there is still a little piece missing.



In a similar fashion to above, we can fit 3 pieces on this ruler, but the pieces have length that is $\frac{1}{10}$ the length of our previous piece, with a bit left over. Adding up the two bits we have so far, we can say that approximately

$$\frac{1}{3} \sim 3 \times \frac{1}{10} + 3 \times \frac{1}{10^2} = 0.3 + 0.03 = 0.33.$$

This process can continue, where next we have a ruler of length 0.1 rather than 1. Here it is to scale with our original ruler (impossible to see!) and then blown up:

$\longleftarrow$ Can you see it?

Adding up the three bits we have so far, we can say that approximately

$$\frac{1}{3} \sim 3 \times \frac{1}{10} + 3 \times \frac{1}{10^2} + 3 \times \frac{1}{10^3} = 0.3 + 0.03 + 0.003 = 0.333.$$

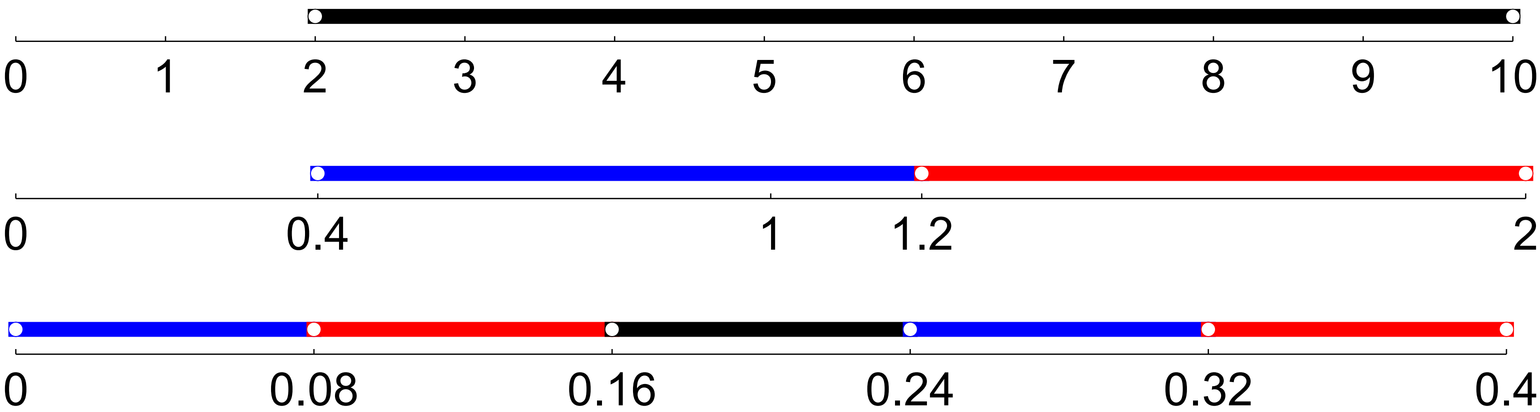
Continuing, this process shows why

$$\frac{1}{3} = 0.33333333333\ldots = 0.\overline{3}. \qquad \text{(the bar means that part of the decimal repeats)}$$

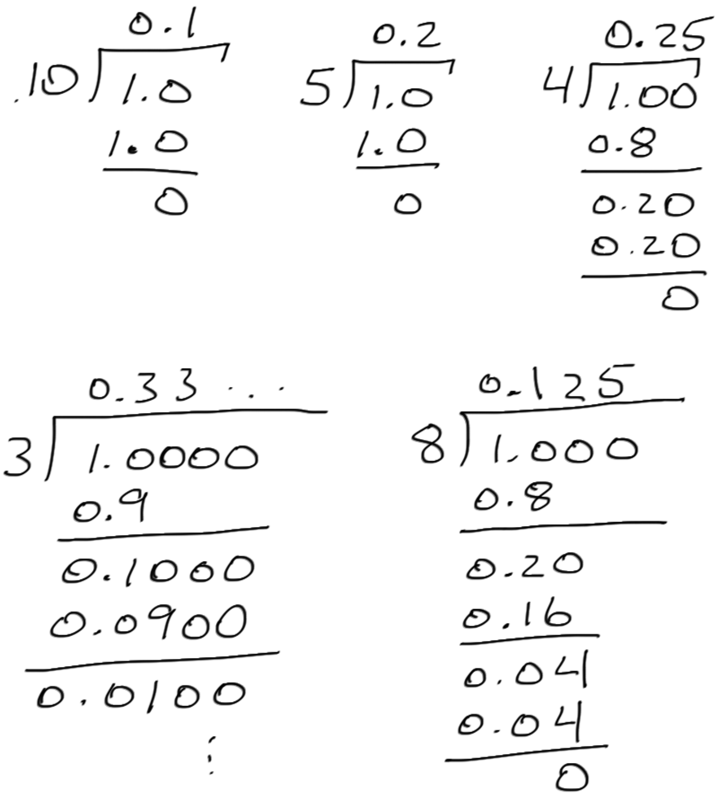
The fraction 1/8 as a decimal

The number lines below show why we can say

$$\frac{1}{8} = 1 \times \frac{1}{10} + 2 \times \frac{1}{10^2} + 5 \times \frac{1}{10^3} = 0.1 + 0.02 + 0.005 = 0.125.$$



Of course, you will most likely use a calculator to convert from fraction to decimal, but it is always nice to have a sense of what is going on mathematically, and the process outlined above is what you are really doing when you do long division to convert a fraction to a decimal.



For practice, see if you can understand verify the following conversions from fraction to decimal:

$$\frac{1}{6} = 0.1\overline{6}, \qquad \frac{1}{7} = 0.\overline{142857}, \qquad \frac{1}{9} = 0.\overline{1}.$$

Prime Numbers

An important number when working with fractions is a **prime number**, which is a natural number that only has itself and 1 as divisors.

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$ first few prime numbers

Notice the number 1 is not considered a prime number.

2.2 Prime Factorization of an Integer

Technique: Find a number that divides evenly, and continue the process until you have a prime factorization:

$$\begin{aligned} 168 &= 2 \times \mathbf{84}, \\ &= 2 \times \mathbf{2 \times 42}, \\ &= 2 \times 2 \times 2 \times 21, \\ &= \mathbf{2 \times 2 \times 2} \times 3 \times 7, \\ &= \mathbf{2^3} \times 3^1 \times 7^1. \end{aligned}$$

Prime factorization is a fundamental tool that is used to work with fractions, where we often need to prime factor the numerator and the denominator.

2.3 Equivalent Fractions

Equivalent fractions have the same value. Since $\frac{1}{2} = \frac{2}{4}$, we can say $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent fractions.

Technique to determine equivalent fraction: Prime factor numerator and denominator, then cancel common factors.

$$\frac{42}{140} = \frac{2 \times 3 \times 7}{2 \times 2 \times 7 \times 5} = \frac{\cancel{2} \times 3 \times \cancel{7}}{\cancel{2} \times 2 \times \cancel{7} \times 5} = \frac{3}{2 \times 5} = \frac{3}{10}.$$

A fraction is in **reduced form** (sometimes called simplest form) when all the common factors are cancelled in the numerator and denominator. So $\frac{3}{10}$ is the simplest form of $\frac{42}{140}$.

To change fractions to equivalent fractions with a specific denominator, prime factor the denominator you want and then multiply the fraction's numerator and denominator by what is missing.

EXAMPLE Convert $\frac{23}{3}$ to have a denominator of 33.

SOLUTION: Since $33 = 3 \times 11$, we have to multiply the denominator by 11 to get 33. We must also multiply the numerator by 11 so the value of the fraction remains the same (we are multiplying by 1, so we retain an equivalent fraction).

$$\frac{23}{3} = \frac{23 \times 11}{3 \times 11} = \frac{253}{33}.$$

2.4 Multiplication of Fractions

Multiply the numerators and denominators, being careful to use parentheses where needed. Note that finding a common denominator is not required to multiply fractions.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}. \qquad \text{(multiply)}$$

In the notation above, a, b, c, d just represent any integer.

EXAMPLE $\frac{5}{14} \times \frac{7}{12} = \frac{5 \times 7}{14 \times 12} = \frac{5 \times \cancel{7}}{2 \times \cancel{7} \times 4 \times 3} = \frac{5}{24}.$

2.5 Division of Fractions

Multiply by the reciprocal of the quantity you are dividing by:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}. \qquad \text{(divide)}$$

EXAMPLE $\frac{1}{4} \div \frac{7}{12} = \frac{1}{4} \times \frac{12}{7} = \frac{1 \times 12}{4 \times 7} = \frac{\cancel{4} \times 3}{\cancel{4} \times 7} = \frac{3}{7}.$

2.6 Adding and Subtracting Fractions

You must have common denominators before you can add or subtract fractions.

$$\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}. \qquad \text{(adding fractions with common denominators)}$$

$$\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}. \qquad \text{(subtracting fractions with common denominators)}$$

2.7 Finding the LCD (least common denominator) of two or more fractions

Technique: Look for the prime factorization of the two denominators. Collect common factors.

EXAMPLE Find LCD of $\frac{3}{16}$ and $\frac{7}{28}$. Then add the fractions.

SOLUTION: Prime factorization of the denominators:

$$16 = 2 \times 2 \times 2 \times 2 \qquad \text{missing } 7$$

$$28 = 2 \times 2 \qquad \times 7 \qquad \text{missing } 2 \times 2 = 4$$

$$\text{LCD} = 2 \times 2 \times 2 \times 2 \times 7 = 2^4 \times 7 = 112$$

You can now use this to add or subtract the fractions:

$$
\begin{aligned}
\frac{3}{16}+\frac{7}{28} &= \frac{3\times\mathbf{7}}{16\times\mathbf{7}}+\frac{7\times\mathbf{4}}{28\times\mathbf{4}}, && \text{(notice what you multiply by are the bits missing in the LCD table!)}\\
&= \frac{21}{112}+\frac{28}{112}, && \text{(now that you have common denominators, you can add the fractions)}\\
&= \frac{21+28}{112},\\
&= \frac{49}{112}, && \text{(now you can prime factor and simplify the fraction)}\\
&= \frac{\cancel{7}\times 7}{\cancel{7}\times 16},\\
&= \frac{7}{16}. && \text{(reduced form)}
\end{aligned}
$$

EXAMPLE Simplify $\frac{1}{5}+\frac{8}{25}-\frac{9}{21}$.

SOLUTION: Prime factorization of the denominators:

$$
\begin{aligned}
5 &= 5\\
25 &= 5\times 5\\
21 &= \qquad\quad 3\times 7\\
\hline
\text{LCD} &= 5\times 5\times 3\times 7 = 5^2\times 3\times 7 = 525
\end{aligned}
$$

You can now use this to add or subtract the fractions:

$$
\begin{aligned}
\frac{1}{5}+\frac{8}{25}-\frac{9}{21} &= \frac{1\times\mathbf{5}\times\mathbf{3}\times\mathbf{7}}{5\times\mathbf{5}\times\mathbf{3}\times\mathbf{7}}+\frac{8\times\mathbf{3}\times\mathbf{7}}{25\times\mathbf{3}\times\mathbf{7}}-\frac{9\times\mathbf{5}\times\mathbf{5}}{21\times\mathbf{5}\times\mathbf{5}}, && \text{(multiply by what is needed to get LCD)}\\
&= \frac{105}{525}+\frac{168}{525}-\frac{225}{525}, && \text{(common denominators, so we can add)}\\
&= \frac{105+168-225}{525}, && \text{(add fractions with common denominators)}\\
&= \frac{48}{525},\\
&= \frac{\cancel{3}\times 16}{5^2\times\cancel{3}\times 7}, && \text{(prime factor)}\\
&= \frac{16}{175}. && \text{(reduced form)}
\end{aligned}
$$

You can add fractions using a denominator that isn't the least common denominator, but the numbers are larger than when you use LCD, so this is not advised.

$$\frac{1}{4}+\frac{3}{8}=\frac{1\times\mathbf{8}}{4\times\mathbf{8}}+\frac{3\times\mathbf{4}}{8\times\mathbf{4}}=\frac{8}{32}+\frac{12}{32}=\frac{8+12}{32}=\frac{20}{32}=\frac{\cancel{4}\times 5}{\cancel{4}\times 8}=\frac{5}{8}.$$

$$\frac{1}{4}+\frac{3}{8}=\frac{1\times\mathbf{2}}{4\times\mathbf{2}}+\frac{3}{8}=\frac{2}{8}+\frac{3}{8}=\frac{2+3}{8}=\frac{5}{8}.$$

To **add or subtract** fractions you must use a common denominator.

To **multiply** fractions does not require an LCD, simply multiply numerators and multiply denominators.

To **divide** fractions does not require an LCD. You invert the fraction you are dividing by and multiply.

Advice: I highly recommend that you get in the habit of using brackets when you have multiple denominators.

$$\frac{1}{4} \div \frac{4}{5} \text{ is written as } \frac{\left(\frac{1}{4}\right)}{\left(\frac{4}{5}\right)} \text{ instead of } \frac{\frac{1}{4}}{\frac{4}{5}}.$$

You will avoid many errors if you do this.

2.8 Negative Numbers

To deal with negative numbers, it can be helpful to think of a factor of -1. Arithmetic of negative numbers is also discussed in Unit 2.

EXAMPLE Factor the negative number -126.

$$-126 = -1 \times 2 \times 3^2 \times 7.$$

Treating the negative as a factor of -1 can be helpful when performing arithmetic with negative numbers, since you can cancel or combine the -1 to get the correct overall sign.

Also note the minus sign can occur in a variety of places. All of these represent the same number:

$$-1 \times \frac{2}{3} = -\frac{2}{3} = \frac{-2}{3} = \frac{2}{-3}.$$

EXAMPLE Simplify $-14/8 \div (-36/9)$.

$$\begin{aligned}
-14/8 \div (-36/9) &= -14/8 \times (-9/36) && \text{(division of fractions by invert and multiply)} \\
&= -\frac{14}{8} \times \left(-\frac{9}{36}\right) && \text{(write fractions stacked)} \\
&= \frac{-14}{8} \times \frac{-9}{36} && \text{(move minus signs to numerator)} \\
&= \frac{(-14) \times (-9)}{8 \times 36} && \text{(multiplication of fractions)} \\
&= \frac{(-1 \times 2 \times 7) \times (-1 \times 3^2)}{2^3 \times 2^2 \times 3^2} && \text{(factor)} \\
&= \frac{(-1)^2 \times 2 \times 7 \times \cancel{3^2}}{2^5 \times \cancel{3^2}} && \text{(collect factors, note } (-1)^2 = 1\text{)} \\
&= \frac{7}{2^4} && \text{(cancel common factors)} \\
&= \frac{7}{16} && \text{(reduced form)}
\end{aligned}$$

There are of course other ways to track the minus signs throughout the solution which are also correct.

2.9 Improper Fractions and Mixed Numbers

An **improper fraction** has a numerator that is larger than the denominator, so $\frac{3}{2}$ is an improper fraction.

A **mixed number** is an improper fraction with the whole number divided out.

EXAMPLE The improper fraction $\frac{25}{6} = 4\frac{1}{6}$ since 6 divides into 25 a total of 4 times with a remainder of 1. Notice that we would read $4\frac{1}{6}$ as "four and one sixth", where the word "and" indicates the addition inherent in a mixed number.

To convert from a mixed number to an improper fraction, write the addition explicitly and then add:

$$4\frac{1}{6} = 4 + \frac{1}{6} = \frac{4}{1} + \frac{1}{6} = \frac{4 \times \mathbf{6}}{1 \times \mathbf{6}} + \frac{1}{6} = \frac{24}{6} + \frac{1}{6} = \frac{24+1}{6} = \frac{25}{6}.$$

You should be comfortable converting between improper fractions and mixed numbers.

To **add, subtract, multiply, or divide mixed numbers:** change to improper fraction.

Note when adding or subtracting (not multiplying or dividing!) mixed numbers you may also add the whole parts and the fractional parts separately, which can be useful when the numbers are large and the arithmetic needed to convert to an improper fraction would take some time to complete.

EXAMPLE Add the mixed numbers $67\frac{3}{32} + 128\frac{7}{16}$.

$$\begin{aligned}
67\frac{3}{32} + 128\frac{7}{16} &= 67 + \frac{3}{32} + 128 + \frac{7}{16} && \text{(write the addition explicitly)}\\
&= 67 + 128 + \frac{3}{32} + \frac{7}{16} && \text{(collect whole numbers and fractions together)}\\
&= 195 + \frac{3}{32} + \frac{7 \times \mathbf{2}}{16 \times \mathbf{2}} && \text{(add whole numbers, get common denominator)}\\
&= 195 + \frac{3}{32} + \frac{14}{32} && \\
&= 195 + \frac{17}{32} && \text{(add fractions)}\\
&= 195\frac{17}{32} && \text{(write result as a mixed number)}
\end{aligned}$$

Improper fractions are used more frequently in higher math than mixed numbers, and in your math classes you should use improper fractions unless there is a specific application where mixed numbers are more appropriate.

It is far more common to see $\frac{4}{3}$ instead of $1\frac{1}{3}$. The reason for this is that when we place two variables together we mean multiplication, $ab = a \times b$, but when we place to numbers together we usually mean addition $2\frac{2}{3} = 2 + \frac{2}{3}$.

Compare the following, the first is multiplication, the second is a mixed number:

$$5\left(\frac{2}{3}\right) = 5 \times \frac{2}{3} = \frac{5}{1} \times \frac{2}{3} = \frac{5 \times 2}{1 \times 3} = \frac{10}{3} \qquad \text{(multiplication)}$$

$$5\frac{2}{3} = 5 + \frac{2}{3} = \frac{15}{3} + \frac{2}{3} = \frac{15+2}{3} = \frac{17}{3} \qquad \text{(mixed number)}$$

To avoid this confusion, use multiplication symbols or brackets (either round or square brackets, round brackets are called parenthesis) when placing two numbers together:

$$5 \times \frac{2}{3} = 5 \cdot \frac{2}{3} = 5\left(\frac{2}{3}\right) = 5\left[\frac{2}{3}\right] \qquad \text{(displaying multiplication clearly)}$$

3 Percents

The percent sign % can be thought of as "per one hundred", i.e., % is equivalent to multiplying by $\frac{1}{100}$.

EXAMPLE Changing a percent to a decimal

$$6.7\% = 6.7 \times \frac{1}{100} = \frac{6.7}{100} = 0.067.$$

Advice: Get in the habit of writing the leading zero when using decimals (0.067 instead of .067) since that makes the number clearer to read and helps avoid errors.

EXAMPLE Changing a decimal to a percent

$$0.00078 = 0.078\%. \qquad \text{(move decimal two places to the left)}$$

EXAMPLE Finding a percent of a number

27.5% of 5.1 is found by the following:

$$27.5\% \times 5.1 = 0.275 \times 5.1 = 1.4025.$$

This makes sense since the answer should be close to 25% of 5 which is 1.25.

EXAMPLE Finding what percent of one number is of another number

What percent of 5 is 160?

$$\frac{160}{5} = 32 = 3200\%.$$

This makes sense since 160 is larger than 5, so it should be greater than 100% of 5.

As your knowledge of mathematics progresses, you will learn different ways of doing things that may be easier.

For example, to change fractions to a new denominator, you can think of the problem as one in which you solve for an unknown x,

$$\frac{23}{3} = \frac{x}{9} \qquad \text{(solve for } x\text{)}$$

$$\frac{23}{3}\times 9 = \frac{x}{9}\times 9 \qquad \text{(algebra: multiply both sides of equation by 9)}$$

$$\frac{23 \times 3 \times \cancel{3}}{\cancel{3}} = \frac{x}{\cancel{9}} \times \cancel{9} \qquad \text{(simplify)}$$

$$69 = x$$

$$\text{so } \frac{23}{3} = \frac{69}{9}$$

4 Dimensional Analysis

When we create an equivalent fraction, we are essentially multiplying by 1 to create an equivalent expression.

$$\frac{1}{3} = \frac{1}{3} \times \mathbf{1} = \frac{1}{3} \times \frac{\mathbf{2}}{\mathbf{2}} = \frac{2}{6}$$

This is the same process used when we want to change the units of a number.

EXAMPLE Change 32 ft to inches.

Since 1 ft = 12 in, we can write the following since $\frac{12 \text{ in}}{1 \text{ ft}} = 1$:

$$\begin{aligned} 32 \text{ ft} &= 32 \text{ ft} \times \mathbf{1} \\ &= 32 \text{ ft} \times \frac{\mathbf{12} \text{ in}}{\mathbf{1} \text{ ft}} \\ &= 32 \cancel{\text{ft}} \times \frac{\mathbf{12} \text{ in}}{\mathbf{1} \cancel{\text{ft}}} \\ &= 384 \text{ in} \end{aligned}$$

EXAMPLE How many minutes are there in 1 year?

$$\begin{aligned} 1 \text{ year} &= 1 \text{ year} \times \frac{\mathbf{365} \text{ days}}{\mathbf{1} \text{ year}} \times \frac{\mathbf{24} \text{ hour}}{\mathbf{1} \text{ day}} \times \frac{\mathbf{60} \text{ minutes}}{\mathbf{1} \text{ hour}} \\ &= 1 \cancel{\text{year}} \times \frac{\mathbf{365} \cancel{\text{days}}}{\mathbf{1} \cancel{\text{year}}} \times \frac{\mathbf{24} \cancel{\text{hour}}}{\mathbf{1} \cancel{\text{day}}} \times \frac{\mathbf{60} \text{ minutes}}{\mathbf{1} \cancel{\text{hour}}} \\ &= 525\,600 \text{ minutes} \end{aligned}$$

EXAMPLE Convert 110 kilometers per hour to miles per hour.

We need to know that 1 kilometer = 0.6214 miles.

$$\begin{aligned} 110 \frac{\text{km}}{\text{hr}} &= 110 \frac{\cancel{\text{km}}}{\text{hr}} \times \frac{\mathbf{0.6214} \text{ miles}}{\mathbf{1} \cancel{\text{km}}} \\ &= 68.35 \frac{\text{miles}}{\text{hr}} \end{aligned}$$

So if you know the unit conversions, you can switch units effectively using this process. Notice that by checking that your units are canceling properly you will ensure you have made the correct conversion.