1 Radical Equations

An equation that has the variable to be solved for inside a radical is called a **radical equation**.

The algebraic manipulations (described below) needed to solve the equation for the variable can be involved, and may result in extraneous solutions.

An **extraneous solution** is an apparent solution that arose through the algebraic manipulations, but is not a solution of the original equation.

Solving radical equations:

1. Algebraically isolate one radical by itself on one side of equal sign.
2. Raise each side of the equation to an appropriate power to remove the radical.
3. Simplify.
4. If the equation sill contains a radical, repeat steps 1 through 3.
5. Once all the radicals are removed, solve the equation.
6. Check all solutions, and exclude any that do not satisfy the original equation. These excluded solutions are called extraneous solutions.

When we wrote the rules of radicals earlier, we wrote them in terms of x. Let's write them again, but now write them in terms of an expression A (or B) that contains x.

All the different cases we saw earlier for how to simplify when x is positive or negative are what can lead to the extraneous solutions. Using the expression A, all the different cases simplify to one case, and the checking to eliminate extraneous solutions at the end is what takes care of A being positive or negative.

So we don't have to worry about if A is positive or negative, as any issues arising from $A < 0$ are taken care of when we eliminate the extraneous solutions.

Simplifying a radical of an expression:

$$\left(\sqrt[n]{A}\right)^n = A \qquad \text{(when the expression } A \text{ contains } x\text{)}$$

$$\sqrt[n]{A}\sqrt[n]{B} = \sqrt[n]{AB} \qquad \text{(product rule for radicals)}$$

You will also need to use other algebraic techniques, such as the addition, multiplication, and division principles (Unit 3) and factoring (Unit 7).

The Grouping Method to factor trinomials of form $ax^2 + bx + c$

1. Determine the grouping number ac.
2. Find two numbers whose product is ac and sum is b.
3. Use these numbers to write bx as the sum of two terms.
4. Factor by grouping.
5. Check your answer by multiplying out.

EXAMPLE Solve $2\sqrt{4x+1}+5=x+9$.

$$
\begin{aligned}
2\sqrt{\mathbf{4x+1}}+5 &= x+9 && \text{(isolate the radical } \sqrt{4x+1}\text{)}\\
2\sqrt{4x+1}+\cancel{5}\mathbf{-\cancel{5}} &= x+9\mathbf{-5} && \text{(addition principle)}\\
\frac{\cancel{2}\sqrt{4x+1}}{\mathbf{\cancel{2}}} &= \frac{x+4}{\mathbf{2}} && \text{(division principle)}\\
\sqrt{4x+1} &= \frac{x+4}{2} && \text{(radical is now isolated)}\\
\left(\sqrt{4x+1}\right)^{\mathbf{2}} &= \left(\frac{x+4}{2}\right)^{\mathbf{2}} && \text{(now square each side to remove radical)}\\
4x+1 &= \frac{(x+4)^2}{4} && \text{(rules of radicals, } \left(\sqrt{A}\right)^2=A\text{)}\\
\mathbf{4\times}(4x+1) &= \mathbf{\cancel{4}\times}\frac{(x+4)^2}{\cancel{4}} && \text{(multiplication principle)}\\
16x+4 &= (x+4)^2 && \text{(quadratic–set it up for factoring)}\\
16x+4 &= x^2+8x+16 && \text{(multiply out power)}\\
x^2-8x+12 &= 0 && \text{(collect like terms)}\\
(x-6)(x-2) &= 0 && \text{(factor)}\\
(x-6)=0 \text{ or } (x-2) &= 0 && \text{(zero factor property)}\\
x=6 \text{ or } x &= 2
\end{aligned}
$$

We are not done until we have checked that these really are solutions.

$$
\begin{aligned}
2\sqrt{4(6)+1}+5 &\stackrel{?}{=} (6)+9 && \text{(check } x=6\text{)}\\
2\sqrt{25}+5 &\stackrel{?}{=} 15\\
15 &\stackrel{?}{=} 15 && \text{(True! } x=6 \text{ is a solution)}
\end{aligned}
$$

$$
\begin{aligned}
2\sqrt{4(2)+1}+5 &\stackrel{?}{=} (2)+9 && \text{(check } x=2\text{)}\\
2\sqrt{9}+5 &\stackrel{?}{=} 11\\
11 &\stackrel{?}{=} 11 && \text{(True! } x=2 \text{ is a solution)}
\end{aligned}
$$

The solution to $2\sqrt{4x+1}+5=x+9$ is $x=2$ and $x=6$.

EXAMPLE Solve $\sqrt{3x+4}+\sqrt{x+5}=\sqrt{7-2x}$.

$$\begin{aligned}
\sqrt{3x+4}+\sqrt{x+5} &= \sqrt{7-2x} && \text{(a radical is isolated, so square both sides)}\\
\left(\sqrt{3x+4}+\sqrt{x+5}\right)^{\mathbf{2}} &= \left(\sqrt{7-2x}\right)^{\mathbf{2}} && \text{(simplify)}\\
\left(\sqrt{3x+4}\right)^2+\left(\sqrt{x+5}\right)^2+2\sqrt{3x+4}\sqrt{x+5} &= 7-2x\\
3x+4+x+5+2\sqrt{3x+4}\sqrt{x+5} &= 7-2x && \text{(collect like terms, product rule for radicals)}\\
\sqrt{(3x+4)(x+5)} &= -1-3x && \text{(isolate the remaining radical)}\\
\left(\sqrt{(3x+4)(x+5)}\right)^{\mathbf{2}} &= (-1-3x)^{\mathbf{2}} && \text{(square both sides to remove the radical)}\\
(3x+4)(x+5) &= 1+6x+9x^2 && \text{(multiply out power)}\\
3x^2+19x+20 &= 1+6x+9x^2 && \text{(collect like terms)}
\end{aligned}$$

Now we have a quadratic to factor. The numbers are large, so this is a challenging one!

Two numbers whose product is $-19\times 6=-114$ sum is -13: -19, 6.

$$\begin{aligned}
6x^2\,\mathbf{-\,13x}-19 &= 0 && \text{(factor by grouping)}\\
6x^2\,\mathbf{-\,19x+6x}-19 &= 0\\
\left(6x^2-19x\right)+\left(6x-19\right) &= 0\\
x\mathbf{(6x-19)}+1\mathbf{(6x-19)} &= 0 && \text{(remove largest common factors)}\\
(x+1)\mathbf{(6x-19)} &= 0 && \text{(factor)}\\
(x+1)=0 \text{ or } (6x-19) &= 0 && \text{(zero factor property)}\\
x=-1 \text{ or } x &= \frac{19}{6} && \text{(potential solutions)}
\end{aligned}$$

We aren't done until we see if these solve the original equation:

$$\begin{aligned}
\sqrt{3(-1)+4}+\sqrt{(-1)+5} &\stackrel{?}{=} \sqrt{7-2(-1)} && \text{(check } x=-1\text{)}\\
\sqrt{1}+\sqrt{4} &\stackrel{?}{=} \sqrt{9}\\
1+2 &\stackrel{?}{=} 3 && \text{(True! So } x=-1 \text{ is a solution)}
\end{aligned}$$

$$\begin{aligned}
\sqrt{3(19/6)+4}+\sqrt{(19/6)+5} &\stackrel{?}{=} \sqrt{7-2(19/6)} && \text{(check } x=19/6\text{)}\\
\sqrt{27/2}+\sqrt{49/6} &\stackrel{?}{=} \sqrt{2/3} && \text{(False! So } x=19/6 \text{ is not a solution)}
\end{aligned}$$

This last step you could use a calculator to check. Sometimes checking to eliminate extraneous solutions can be a tremendous amount of work, so recognize when a calculator would be a useful tool to assist.

The only solution to $\sqrt{3x+4}+\sqrt{x+5}=\sqrt{7-2x}$ is $x=-1$.

2 Complex numbers

There is a another kind of number that is not a real number. The new number is called a **complex number**.

Complex numbers are incredibly useful in many branches of science. Here we just want you to be comfortable with the arithmetic of complex numbers, and see that complex numbers are what we obtain when we take a square root of a negative number.

Complex Number: $a + bi$

- the real number a is the **real part** of the complex number,
- the real number b is the **imaginary part** of the complex number,
- the quantity i is defined as $i = \sqrt{-1}$, or $i^2 = -1$.

A complex number with the form bi (where $a = 0$) is called an **imaginary number**. The phrase "imaginary number" is a bit misleading–the number exists, it is just something other than a real number!

The **complex conjugate** of the number $a + bi$ is defined as $a - bi$ (i.e., change the sign of the imaginary part).

Complex numbers are incredibly useful in a variety of areas of mathematics. At the moment, we are interested in them since the arise when taking the square root of a negative number, so

$$\begin{aligned}
\sqrt{4} &= 2 \text{ is a real number.}\\
\sqrt{-4} &= \sqrt{4}\sqrt{-1} = 2i \text{ is a complex number. Since there is no real part, this is an imaginary number.}\\
7 + \sqrt{25} &= 7 + 5 = 12 \text{ is a real number.}\\
7 + \sqrt{-25} &= 7 + \sqrt{25}\sqrt{-1} = 7 + 5i \text{ is a complex number.}\\
\sqrt{3} & \text{ is a real number.}\\
\sqrt{-3} &= \sqrt{3}\sqrt{-1} = \sqrt{3} \cdot i = i\sqrt{3} \text{ is an imaginary number, since there is no real part.}
\end{aligned}$$

EXAMPLE Is it true that $\sqrt{(-4)(-4)} = \sqrt{(-4)}\sqrt{(-4)}$? Investigate!

This should not be true, since the product rule for radicals says $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ is a and b are nonegative real numbers (see Unit 10).

$$\begin{aligned}
\sqrt{(-4)(-4)} &= \sqrt{16} = 4\\
\sqrt{-4}\sqrt{-4} &= 2i \cdot 2i\\
&= 4i^2\\
&= 4(-1) && \text{(since } i^2 = -1\text{)}\\
&= -4
\end{aligned}$$

So $\sqrt{(-4)(-4)} \neq \sqrt{(-4)}\sqrt{(-4)}$, as we expected. We still have to be careful when using the rules of radicals.

For now, you should see that the real parts are collected together and the imaginary parts are collected together when you are working with complex numbers. The basic fact that $i^2 = -1$ allows you to simplify complex numbers.

2.1 Arithmetic of Complex Numbers

Understand the process, and just work it out when you need to. Do not memorize the formulas.

Addition or Subtraction: Collect real part together and imaginary parts together.

$$(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$$

Multiplication: Use the distributive property and the fact that $i^2 = -1$ to simplify.

$$(a+bi) \times (c+di) = (ac-bd) + (ad+bc)i$$

Division: Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(a+bi)}{(c+di)} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

EXAMPLE Add the complex numbers $3-7i$ and $-2+3i$.

$$\begin{aligned} (3-7i)+(-2+3i) &= (3-2)+(-7i+3i) && \text{(combine real parts together and imaginary parts together)} \\ &= 1-4i \end{aligned}$$

EXAMPLE Multiply the complex numbers $3-7i$ and $-2+3i$.

$$\begin{aligned} (3-7i) \times \mathbf{(-2+3i)} &= 3\mathbf{(-2+3i)} - 7i\mathbf{(-2+3i)} && \text{(distribute)} \\ &= -6+9i+14i-21\boldsymbol{i^2} && \text{(distribute)} \\ &= -6+9i+14i-21\mathbf{(-1)} && \text{(replace } i^2=-1\text{)} \\ &= \mathbf{-6+9i+14i+21} && \text{(collect real an imaginary part together)} \\ &= 15+23i \end{aligned}$$

EXAMPLE Divide the complex number $3-7i$ by $-2+3i$.

$$\begin{aligned} (3-7i) \div (-2+3i) &= \frac{3-7i}{-2+3i} \times \frac{\mathbf{-2-3i}}{\mathbf{-2-3i}} && \text{(complex conjugate of } -2+3i \text{ is } -2-3i\text{)} \\ &= \frac{(3-7i)(-2-3i)}{(-2+3i)(-2-3i)} \\ &= \frac{-6-9i+14i+21\boldsymbol{i^2}}{4+6i-6i-9\boldsymbol{i^2}} && \text{(distribute)} \\ &= \frac{-6+5i+21\mathbf{(-1)}}{4+\cancel{6i}-\cancel{6i}-9\mathbf{(-1)}} && \text{(collect like terms, replace } i^2=-1\text{)} \\ &= \frac{-27+5i}{13} && \text{(collect like terms)} \\ &= -\frac{27}{13}+\frac{5}{13}i \end{aligned}$$

Example Divide the two complex numbers: $\dfrac{7+14i}{6-3i}$.

$$\begin{aligned}
\frac{7+14i}{6-3i} &= \frac{7+14i}{6-3i} \times \frac{\mathbf{6+3}\boldsymbol{i}}{\mathbf{6+3}\boldsymbol{i}} && \text{(complex conjugate of } 6+3i \text{ is } 6-3i\text{)}\\
&= \frac{(7+14i)(6+3i)}{(6-3i)(6+3i)}\\
&= \frac{42+84i+21i+42\boldsymbol{i^2}}{36-9\boldsymbol{i^2}} && \text{(distribute)}\\
&= \frac{42+105i+42\mathbf{(-1)}}{36-9\mathbf{(-1)}} && \text{(use } i^2=-1\text{)}\\
&= \frac{\cancel{42}+105i-\cancel{42}}{36+9}\\
&= \frac{105i}{45} && \text{(simplify)}\\
&= \frac{\cancel{15}\times 7i}{\cancel{15}\times 3} && \text{(prime factor)}\\
&= \frac{7}{3}i && \text{(reduced form)}
\end{aligned}$$

3 Variation

Variation is an important way to build formulas in the sciences, when how quantities are related are known.

- **Direct variation** between x and y means $y = kx$, where k is the constant of variation.
- **Inverse variation** between x and y means $y = \dfrac{k}{x}$, where k is the constant of variation.
- In either case, use the given information to determine the value of k.
- Then, use the equation you have created to determine the unknown quantity.
- **Joint variation** just means a quantity depends on the product of more than one quantity.

To answer variation problems:

1. Use the given information to determine a relationship between the quantities, which includes a constant of variation k. You may have to define some variables and include units at this step.
2. Use the data given to determine k.
3. Substitute the value of k you found into your relationship.
4. You now have a formula that relates the quantities in the problem, which will allow you to answer the question.

EXAMPLE Police officers can use variation to detect speeding. The speed of a car varies inversely with the time it takes to cover a certain fixed distance. Between two points on a highway, a car travels 45 mph in 6 seconds. What is the speed of a car that travels the same distance in 9 seconds?

Step 1: Get Relationship. We are told there is an inverse variation between speed and time:

$$v = \frac{k}{t},$$

where t is time in seconds and v is velocity in mph.

Step 2: Get value for constant of variation. Use the given information (car traveling at 45 mph covers the distance in 6 seconds) to determine the value of the constant of variation k:

$$v = \frac{k}{t} \quad \Rightarrow \quad 45 = \frac{k}{6} \quad \Rightarrow \quad k = 270.$$

Step 3: Get Formula. Now we have a formula for the speed of the car:

$$v = \frac{270}{t}.$$

Step 3: Answer Question. Now we can answer the question asked about the speed of a car that covers the distance in 9 seconds:

$$v = \frac{270}{t} \quad \Rightarrow \quad v = \frac{270}{9} = 30 \text{ mph}.$$

Another interesting question to ask is: If the speed limit is 65 mph, what time will a speeding car cover the distance?

$$v = \frac{270}{t} \quad \Rightarrow \quad 65 = \frac{270}{t} \quad \Rightarrow \quad t = 4.15 \text{ seconds}.$$

Any car that covers the distance faster than 4.15 seconds is speeding.

EXAMPLE Determine how w varies with $s, t,$ and r given the literal equation $st^{1/6} = 16\sqrt{rw}$.

SOLUTION: Let's solve for w, then read off the variations. Note this is a joint variation since w will depend on s, t, r all at the same time.

$$st^{1/6} = 16\sqrt{rw} \qquad \text{(original equation)}$$

$$\frac{st^{1/6}}{16} = \sqrt{rw} \qquad \text{(Division Principle)}$$

$$\sqrt{rw} = \frac{st^{1/6}}{16} \qquad \text{(change order so } w \text{ term is on left)}$$

$$rw = \left(\frac{st^{1/6}}{16}\right)^2 \qquad \text{(square both sides of equation)}$$

$$rw = \frac{s^2t^{1/3}}{256} \qquad \text{(rules of exponents)}$$

$$w = \frac{s^2t^{1/3}}{256r} \qquad \text{(Division Principle)}$$

Now we can see that

- w varies directly with the square of s
- w varies directly with the cube root of t
- w varies inversely with r

4 Solving Absolute Value Equalities and Inequalities: Three Cases

Case 1: For equalities of the form $\left|ax+b\right| = \left|cx+d\right|$, the solution is

$$ax+b = cx+d \qquad \text{or} \qquad ax+b = -(cx+d).$$

The solution will be two distinct numbers.

Case 2: For inequalities of the form $\left|ax+b\right| < c$, where $c > 0$ the solution is

$$-c < ax+b < c.$$

NOTE: it is important that the $-c$ is on the left and the c is on the right. If this isn't the case, you will get the wrong solution.
The solution will be a set of points between two numbers.

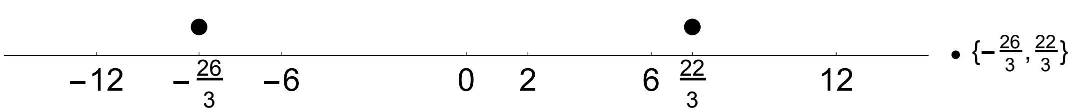
Case 3: For inequalities of the form $\left|ax+b\right| > c$, where $c > 0$ the solution is

$$ax+b < -c \qquad \text{or} \qquad ax+b > c.$$

The solution will be a set of numbers less than one number or greater than another number.

EXAMPLE Solve the inequality $\left|3x+2\right| = 24$ and sketch the solution on a number line.

$$\begin{aligned} \left|3x+2\right| &= 24 \\ 3x+2 = 24 \quad &\text{or} \quad 3x+2 = -24 \qquad \text{(equivalent equations)} \\ 3x = 22 \quad &\text{or} \quad 3x = -26 \\ x = \frac{22}{3} \quad &\text{or} \quad x = -\frac{26}{3} \end{aligned}$$



EXAMPLE Solve the inequality $\left|3x+2\right| < 24$ and sketch the solution on a number line.

$$\left|3x+2\right| < 24$$

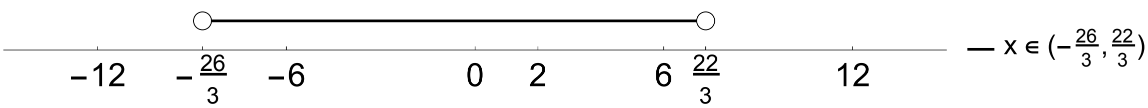
$$-24 < 3x+2 < 24 \qquad \text{(equivalent inequalities)}$$

$$-24 \mathbf{-2} < 3x+\cancel{2} \mathbf{-\cancel{2}} < 24 \mathbf{-2} \qquad \text{(addition principle)}$$

$$-26 < 3x < 22$$

$$\frac{-26}{\mathbf{3}} < \frac{\cancel{3}x}{\mathbf{\cancel{3}}} < \frac{22}{\mathbf{3}} \qquad \text{(division principle)}$$

$$-\frac{26}{3} < x < \frac{22}{3}$$



EXAMPLE Solve the inequality $\left|3x+2\right| > 24$ and sketch the solution on a number line.

$$\left|3x+2\right| > 24$$

$$3x+2 > 24 \quad \text{or} \quad 3x+2 < -24 \qquad \text{(equivalent inequalities)}$$

$$3x > 22 \quad \text{or} \quad 3x < -26$$

$$x > \frac{22}{3} \quad \text{or} \quad x < -\frac{26}{3}$$