1 Equivalent Equations (transposing equations)

We saw earlier that **equivalent expressions** are expressions that represent the same quantity. Equivalent expressions look like

$$\begin{aligned} \text{original expression} &= \text{equivalent expression 1}, \\ &= \text{equivalent expression 2}, \\ &= \text{equivalent expression 3}, \\ &= \text{equivalent expression 4, etc.} \end{aligned}$$

An **equation** involves an equal sign and indicates that two expressions have the same value.

$x + 42 = 67(4 - x)$ is an equation, and means $x + 42$ has the same value as $67(4 - x)$.

Equivalent equations are equations that have *exactly* the same solution.

Transposing an equation involves using the rules of algebra to construct a series of equivalent equations until you determine a numerical solution for an unknown variable, or express one variable in terms of other variables.

To show a set of equivalent equations, we usually rewrite the equivalent equation below the previous equation,

$$\begin{aligned} \text{LHS} &= \text{RHS}, && \text{(original equation)} \\ \text{new LHS} &= \text{new RHS}, && \text{(new equivalent equation)} \\ \text{new LHS} &= \text{new RHS}, && \text{(another equivalent equation)} \end{aligned}$$

The Addition Principle: Add same expression to both sides of an equation. If the same number is added to both sides of an equation, the results on both sides are equal in value (you have constructed an equivalent equation).

$$\begin{aligned} x + 42 &= 67(4 - x) && \text{(an equation)} \\ x + 42 \ \mathbf{- 42} &= 67(4 - x) \ \mathbf{- 42} && \text{(an equivalent equation)} \end{aligned}$$

The Multiplication Principle: Multiply both sides of an equation by same expression. If both sides of an equation are multiplied by the same nonzero number, the results on both sides are equal in value (you have constructed an equivalent equation).

$$\begin{aligned} \frac{1}{12}x + 42 &= 67(4 - y) + 6 && \text{(an equation)} \\ \mathbf{12\times}\frac{1}{12}x &= \mathbf{12\times}\Big(67(4 - y) + 6\Big) && \text{(an equivalent equation)} \end{aligned}$$

The Division Principle: Divide both sides of an equation by same expression. If both sides of an equation are divided by the same nonzero number, the results on both sides are equal in value (you have constructed an equivalent equation).

$$\begin{aligned} 42x &= 67(4 - y) + 6 && \text{(an equation)} \\ \frac{42x}{\mathbf{42}} &= \frac{67(4 - y) + 6}{\mathbf{42}} && \text{(an equivalent equation)} \end{aligned}$$

Note that you have to be careful with division and multiplication. Avoid division by zero, and make sure you multiply or divide each entire side of the equation,

$$x + 42 = 1 - x \qquad \text{(an equation)}$$
$$\mathbf{132}\times x + 42 = \mathbf{132}\times 1 - x \qquad \text{(not an equivalent equation)}$$
$$x + 42 \div \mathbf{69} = 1 - x \div \mathbf{69} \qquad \text{(not an equivalent equation)}$$

The problem with the above is that you are not multiplying or dividing all of the terms on the each side, and thus you have changed the equation. Use parentheses and you will be ok:

$$x + 42 = 1 - x \qquad \text{(an equation)}$$
$$\mathbf{132}(x + 42) = \mathbf{132}(1 - x) \qquad \text{(an equivalent equation)}$$
$$\frac{x + 42}{\mathbf{69}} = \frac{1 - x}{\mathbf{69}} \qquad \text{(an equivalent equation)}$$

1.1 Solving an equation of the form $ax + b = cx + d$

To solve these equations (or even slightly more complicated equations) involves constructing a series of equivalent equations that ends with the equivalent equation "$x =$ solution". The following steps are required, although steps 1 and 2 are only required for the complicated situations.

1. If you don't want to work with fractions, multiply both sides of the equation by the LCD (lowest common denominator).
2. Clear any parentheses using **Distribution Property**.
3. Isolate the variable term on one side of the equal sign using the **Addition Principle**, and collect like terms.
4. Use the **Division Principle** to isolate the variable.

You can check your answer by substituting back in the original equation to see if your answer is correct.

EXAMPLE Solve $3x + 5 = -4x + 2$ for x.

This is a simple case (steps 1 and 2 have nothing to do), so we can start at step 3.

$$3x \cancel{+5} \cancel{\mathbf{-5}} \mathbf{+ 4x} = \cancel{-4x} + 2 \ \mathbf{-5} \cancel{\mathbf{+4x}} \qquad \text{(addition principle)}$$
$$7x = -3 \qquad \text{(cancel and collect like terms)}$$
$$\frac{\cancel{7}x}{\cancel{7}} = \frac{-3}{7} \qquad \text{(division principle)}$$
$$x = -\frac{3}{7} \qquad \text{(cancel common factors)}$$

EXAMPLE Solve for x in $4x + 2 = -7x + 3$.

$$4x + \cancel{2} \mathbf{+ 7x} \cancel{\mathbf{-2}} = \cancel{-7x} + 3 \cancel{\mathbf{+7x}} \mathbf{-2}, \qquad \text{(addition principle)}$$
$$11x = 1, \qquad \text{(collect like terms)}$$
$$x = \frac{1}{11}. \qquad \text{(division principle)}$$

EXAMPLE Solve $\frac{2}{3}(x+4) = 6 - \frac{1}{4}(3x-2) - 1$.

This is one that is a little more complicated, so we will have something to do for steps 1 and 2.

Remember, you might choose a different route to the solution that is entirely correct.

The goal is first to isolate a single term with x in it on one side of the equation.

$$\frac{2}{3}(x+4) = 6 - \frac{1}{4}(3x-2) - 1$$

$$\frac{2}{3}x + \frac{2}{3}\times 4 = 6 - \frac{1}{4}\times 3x - 2\left(-\frac{1}{4}\right) - 1 \quad \text{(I choose to clear parentheses first)}$$

$$\frac{2}{3}x + \frac{8}{3} = 6 - \frac{3}{4}x + \frac{2}{4} - 1 \quad \text{(simplify)}$$

$$\frac{2}{3}x + \frac{8}{3} = 6 - \frac{3}{4}x + \frac{2}{4} - 1 \quad \text{(simplify on each side of equal side by collecting like terms)}$$

$$\frac{2}{3}x + \frac{8}{3} = \frac{11}{2} - \frac{3}{4}x$$

Now use Addition Principle to move all terms with x to left side, all terms with constants to right side.

$$\frac{2}{3}x + \frac{3}{4}x + \cancel{\frac{8}{3}} - \cancel{\frac{8}{3}} = \frac{11}{2} - \frac{8}{3} - \cancel{\frac{3}{4}x} + \cancel{\frac{3}{4}x}$$

$$\frac{2}{3}x + \frac{3}{4}x = \frac{11}{2} - \frac{8}{3} \quad \text{(now collect like terms, get common denominators)}$$

$$\frac{17}{12}x = \frac{17}{6} \quad \text{(now use Multiplication Principle to isolate the } x\text{)}$$

$$\frac{\cancel{12}}{\cancel{17}} \times \frac{\cancel{17}}{\cancel{12}}x = \frac{12}{17} \times \frac{17}{6} \quad \text{(simplify fraction)}$$

$$x = 2$$

Here is an alternate solution, where I multiply by the LCD = 12 first so we don't have to work with fractions.

$$12\times\frac{2}{3}(x+4) = 12\times 6 - 12\times\frac{1}{4}(3x-2) - 12\times 1 \quad \text{(multiplication property)}$$

$$8(x+4) = 72 - 3(3x-2) - 12 \quad \text{(simplify, arithmetic)}$$

$$8x + 32 = 72 - 9x + 6 - 12 \quad \text{(distribution property to clear parenthesis)}$$

$$8x + 32 = -9x + 66 \quad \text{(collect any like terms on each side)}$$

$$8x \cancel{+32} + 9x \cancel{-32} = \cancel{-9x} + 66 \cancel{+9x} - 32 \quad \text{(addition principle to move all } x \text{ to left)}$$

$$17x = 34 \quad \text{(collect like terms)}$$

$$\frac{\cancel{17}x}{\cancel{17}} = \frac{34}{17} \quad \text{(division principle to isolate } x\text{)}$$

$$x = 2 \quad \text{(simplify)}$$

Sometimes when solving an equation strange things can happen.

If you find a true result (like $0 = 0$), then the equation is true for all x (infinite number of solutions).

If you find a false result (like $0 = 4$), then the equation has no solution.

1.2 Canceling Common Factors

If we think of the distribution property in reverse, we can begin to think about what it means to **factor**, which we will study for more complicated situations later.

$$\boldsymbol{a}(b+c) = \boldsymbol{a}b + \boldsymbol{a}c \quad \text{(distribution property)}$$
$$\boldsymbol{a}b + \boldsymbol{a}c = \boldsymbol{a}(b+c) \quad \text{(factoring by common factor)}$$
$$m\boldsymbol{x} + 2\boldsymbol{x} = (m+2)\boldsymbol{x} \quad \text{(factoring common factor, which is combining like terms when there are variables)}$$

A very useful technique when creating equivalent equations is to be able to cancel common factors in the numerator and denominator, so let's look at that here. We will return to this idea when we look at **algebraic fractions** later.

Here are some examples of cancelling a common factor, which you see is the same process whether working with integers or expressions.

$$\frac{\cancel{2} \times 4}{\cancel{2} \times 5} = \frac{4}{5} \quad \text{(cancel common factor of 2)}$$
$$\frac{\cancel{a} \times b}{\cancel{a} \times c} = \frac{b}{c} \quad \text{(cancel common factor of a)}$$
$$\frac{\cancel{P_2 V_2} V_2}{\cancel{P_2 V_2}} = \frac{V_2}{1} = V_2 \quad \text{(cancel common factor of } P_2V_2\text{)}$$
$$\frac{\cancel{2(x-1)}y}{\cancel{2(x-1)}z} = \frac{y}{z} \quad \text{(cancel common factor of } 2(x-1)\text{)}$$

When we are cancelling a common factor, we have to assume the factor does not equal zero, since otherwise we would have had a division by zero! This is a detail we will consider later, but it won't trouble us much.

Often, we don't need to factor, but we do need to cancel common factors when solving equations.

When cancelling common factors:

- Make sure that you are are cancelling something that is a factor of the entire numerator and a factor of the entire denominator.
- Take your time and examine your simplifications to make sure they are correct.

EXAMPLE When you cancel all the factors in the denominator, you still have a one left, not a zero, so a division by zero is a clue that something might have gone wrong:

$$\frac{4^2 \times 5}{4} = \frac{4 \times 5}{1} \quad \text{(correct cancellation of factor 4)}$$
$$\frac{4^2 \times 5}{4} \neq \frac{4 \times 5}{0} \quad \text{(incorrect cancellation of factor 4)}$$
$$\frac{4^2 \times 5}{4} = \frac{\cancel{4} \times 4 \times 5}{1 \times \cancel{4}} = \frac{4 \times 5}{1} \quad \text{(details show why)}$$

EXAMPLE Solve for x in $ax + b = cx + d$.

Note we use the same algebraic process as above, but factor when we come to like terms rather than simply adding them.

$$\begin{aligned}
ax + b &= cx + d, && \text{(move all terms with } x \text{ to left, all terms without } x \text{ to right)}\\
ax + \cancel{b} \boldsymbol{- cx - \cancel{b}} &= \cancel{cx} + d \boldsymbol{- \cancel{cx} - b}, && \text{(addition principle)}\\
a\boldsymbol{x} - c\boldsymbol{x} &= d - b, && \text{(factor } x \text{ on right)}\\
(a - c)\boldsymbol{x} &= d - b, &&\\
\frac{\cancel{(a-c)}x}{\cancel{\boldsymbol{a-c}}} &= \frac{d-b}{\boldsymbol{a-c}}, && \text{(division principle, cancel factor } (a-c))\\
x &= \frac{d-b}{a-c}. &&
\end{aligned}$$

Generally in more complicated examples you try to isolate factors, or isolate terms, on one side.

1.3 Literal Equations

Literal Equations have many unspecified variables, but you solve them using the same techniques. You just can't simplify as much since you are working with variables instead of numbers.

The Combined Gas Law A nice example of a literal equation used in chemistry is the Combined Gas Law, which states that for a gas under two different sets of conditions (labeled by the subscript 1 or 2), it is true that

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}.$$

EXAMPLE An ideal gas in state 1 has $P_1 = 3$ Pa, $V_1 = 20$ cm^3, and $T_1 = 40$ K. This gas is then adjusted so the pressure is $P_2 = 4$ Pa and the volume is $V_2 = 50$ cm^3. What is the temperature of the gas in state 2, using the Combined Gas Law?

$$\begin{aligned}
\frac{P_1V_1}{T_1} &= \frac{P_2V_2}{T_2} && \text{(write the equation you will start with)}\\
\frac{(3\text{ Pa})(20\text{ cm}^3)}{(40\text{ K})} &= \frac{(4\text{ Pa})(50\text{ cm}^3)}{T_2} && \text{(substitute in the values)}\\
T_2 &= \frac{(4\text{ Pa})(50\text{ cm}^3)}{(3\text{ Pa})(20\text{ cm}^3)}(40\text{ K}) && \text{(solve for } T_2)\\
T_2 &= \frac{(4\cancel{\text{ Pa}})(50\cancel{\text{ cm}^3})}{(3\cancel{\text{ Pa}})(20\cancel{\text{ cm}^3})}(40\text{ K}) && \text{(solve for } T_2\text{, cancel units)}\\
T_2 &= \frac{(4)(50)}{(3)(20)}(40)\text{ K} = \frac{400}{3}\text{ K} \sim 133\text{ K} &&
\end{aligned}$$

I split the canceling of units into it's own step, but it need not be. Show as much detail as you need to get the simplification done correctly.

Advice If you need to solve a literal equation for a variable and substitute in values, it is often best to solve for the variable before you substitute in the values.

EXAMPLE An ideal gas in state 1 has $P_1 = 2$ Pa, $V_1 = 20$ cm^3, and $T_1 = 12$ K. This gas is then adjusted so the temperature is $T_2 = 80$ K and the volume is $V_2 = 10$ cm^3. What is the pressure of the gas in state 2, using the Combined Gas Law?

First, solve for P_2.

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$

$$\frac{P_1V_1}{T_1} \cdot \frac{\boldsymbol{T_2}}{\boldsymbol{V_2}} = \frac{P_2\cancel{V_2}}{\cancel{T_2}} \cdot \frac{\cancel{\boldsymbol{T_2}}}{\cancel{\boldsymbol{V_2}}} \qquad \text{(multiplication property)}$$

$$P_2 = \frac{P_1V_1T_2}{T_1V_2} \qquad \text{(multiplication property)}$$

$$P_2 = \frac{(2\text{ Pa})(20\text{ cm}^3))(80\text{ K})}{(12\text{ K})(10\text{ cm}^3)} \qquad \text{(now sub in values)}$$

$$P_2 = \frac{80}{3}\text{ Pa} \sim 27\text{ Pa} \qquad \text{(simplify)}$$

EXAMPLE Solve the Combined Gas Law for T_2.

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2} \qquad \text{(write the equation you will start with)}$$

$$\frac{P_1V_1}{T_1} \times \boldsymbol{T_2} = \frac{P_2V_2}{\cancel{T_2}} \times \cancel{\boldsymbol{T_2}} \qquad \text{(multiplication principle)}$$

$$\frac{\cancel{\boldsymbol{T_1}}}{\cancel{\boldsymbol{P_1V_1}}} \times \frac{\cancel{P_1V_1}}{\cancel{T_1}} \times T_2 = \frac{\boldsymbol{T_1}}{\boldsymbol{P_1V_1}} \times P_2V_2 \qquad \text{(multiplication \& division principle)}$$

$$T_2 = \frac{T_1P_2V_2}{P_1V_1} \qquad \text{(an equivalent equation solved for } T_2\text{)}$$

Here are some more involved examples, just so you can see that even these ones are relying on the addition, multiplication, and division principles to isolate a variable. When it comes to algebra, using these three foundational principles correctly allows you to do some impressive things!

EXAMPLE Solve $A = lxw + lsh + sxw$ for w.

$$A = \boldsymbol{lxw} + lsh + \boldsymbol{sxw} \qquad \text{(identify all the terms with } w \text{ in them)}$$

$$A \boldsymbol{- lsh} = \boldsymbol{lxw} \cancel{+ lsh} + \boldsymbol{sxw} \cancel{\boldsymbol{- lsh}} \qquad \text{(multiplication principle to isolate terms with } w\text{)}$$

$$A - lsh = lx\boldsymbol{w} + sx\boldsymbol{w} \qquad \text{(identify common factor } w\text{)}$$

$$A - lsh = (lx + sx)\boldsymbol{w} \qquad \text{(common factor } w\text{)}$$

$$\frac{A - lsh}{\boldsymbol{(lx + sx)}} = \frac{\cancel{(lx + sx)}w}{\cancel{\boldsymbol{(lx + sx)}}} \qquad \text{(division principle)}$$

$$\frac{A - lsh}{\boldsymbol{(lx + sx)}} = w \qquad \text{(an equivalent equation solved for } w\text{)}$$

EXAMPLE Solve the Svedberg equation $W = \dfrac{RTs}{D\mathbf{(1-vp)}}$ for p. Note p is in a factor (in blue) in denominator, so we will isolate that factor first.

$$W \cdot \mathbf{(1-vp)} = \frac{RTs}{D\cancel{(1-vp)}} \cdot \cancel{\mathbf{(1-vp)}} \quad \text{(multiply each side by factor } (1-vp) \text{ to get } p \text{ in numerator)}$$

$$\mathbf{W} \cdot (1-vp) = \frac{RTs}{D}$$

$$\mathbf{W} - \mathbf{W}vp = \frac{RTs}{D} \quad \text{(distribute } W \text{ into } (1-vp) \text{ to create a term with } p\text{)}$$

$$\cancel{W} - Wvp \mathbf{-\cancel{W}} = \frac{RTs}{D} \mathbf{-W} \quad \text{(subtract } W \text{ from each side to isolate term with } p\text{)}$$

$$\mathbf{-1}(-Wvp) = \mathbf{-1}\left(\frac{RTs}{D} - W\right) \quad \text{(multiply each side by } -1\text{)}$$

$$Wvp = -\frac{RTs}{D} + W \quad \text{(distribute the } -1\text{)}$$

$$Wvp = -\frac{RTs}{D} + \frac{W\mathbf{D}}{\mathbf{D}} \quad \text{(common denominator, use multiplication property)}$$

$$Wvp = \frac{WD - RTs}{D} \quad \text{(simplify)}$$

$$\frac{\cancel{Wv}p}{\cancel{\mathbf{Wv}}} = \frac{WD - RTs}{D\mathbf{Wv}} \quad \text{(divide each side by } Wv \text{ to isolate } p\text{)}$$

$$p = \frac{WD - RTs}{DWv} \quad \text{(simplify, cancel common fators)}$$

EXAMPLE Solve the van der Waals equation (used to model fluid compression in chemistry) for p.

$$\left(p + \frac{n^2a}{V^2}\right)(v - nb) = nRT \quad \text{(van der Waals equation)}$$

$$\left(p + \frac{n^2a}{V^2}\right)\cancel{(v-nb)} \times \frac{\mathbf{1}}{\cancel{\mathbf{(v-nb)}}} = nRT \times \frac{\mathbf{1}}{\mathbf{(v-nb)}} \quad \text{(division principle)}$$

$$p + \frac{n^2a}{V^2} = \frac{nRT}{(v-nb)} \quad \text{(simplify)}$$

$$p + \cancel{\frac{n^2a}{V^2}} \mathbf{-}\cancel{\frac{\mathbf{n^2a}}{\mathbf{V^2}}} = \frac{nRT}{(v-nb)} \mathbf{-\frac{n^2a}{V^2}} \quad \text{(addition principle)}$$

$$p = \frac{nRT}{(v-nb)} - \frac{n^2a}{V^2} \quad \text{(simplify)}$$

Note: Remember, other paths to this result are possible!

2 Solving Linear Inequalities

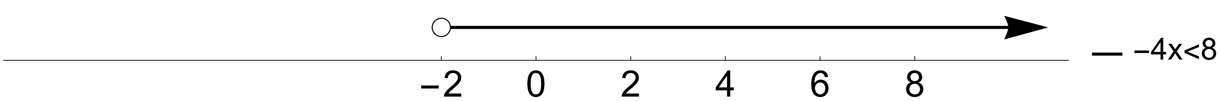
> The process of solution of an inequality is the same as for an equation, except that the inequality is reversed if you multiply or divide by a negative number.

The reason we have to reverse the sign makes sense if we think about it a little bit, using a specific example.

$$
\begin{aligned}
6 &> 4 && \text{(this is true)}\\
-6 &> -4 && \text{(if we multiply by a negative and don't change the direction, the statement is false!)}\\
-6 &< -4 && \text{(if we multiply by a negative and change the direction, the statement is true)}
\end{aligned}
$$

EXAMPLE Show the points satisfying $-4x < 8$ on a number line, and in interval and set notation.

$$
\begin{aligned}
-4x &< 8 && \text{(inequality)}\\
x &> \frac{8}{-4} && \text{(change direction of inequality since divided by a negative)}\\
x &> -2 && \text{(equivalent inequality in set notation)}\\
x &\in (-2, \infty) && \text{(equivalent inequality in interval notation)}
\end{aligned}
$$



EXAMPLE Solve $\dfrac{3x+5}{4} + \dfrac{7}{12} > -\dfrac{x}{6}$.

Let's start this one by clearing fractions. Notice this is similar to solving a more complicated equation where we have to deal with fractions.

So we use the multiplication principle and multiply by the LCD = 12. Since 12 is positive, we don't change the direction of the inequality.

$$
\begin{aligned}
\mathbf{12\times}\left(\frac{3x+5}{4} + \frac{7}{12}\right) &> \mathbf{12\times}\left(-\frac{x}{6}\right) && \text{(now distribute the factor of 12)}\\
\mathbf{12\times}\frac{3x+5}{4} + \cancel{\mathbf{12}}\mathbf{\times}\frac{7}{\cancel{12}} &> -2x &&\\
\mathbf{3(3x+5)} + 7 &> -2x && \text{(distribute the 3)}\\
\mathbf{9x+15} + 7 &> -2x &&\\
9x + 22 &> -2x && \text{(collect like terms)}\\
\cancel{9x} + 22 \cancel{\mathbf{-9x}} &> -2x \mathbf{-9x} && \text{(Use addition principle principle to get all } x \text{ terms on one side)}\\
22 &> -11x && \text{(add like terms)}
\end{aligned}
$$

I decided to subtract $9x$ from both sides just to show you what happens, and that as long as we follow the rules we will get the right answer.

The next step is to use the division principle and divide both sides by -11, but remember that the direction of the inequality changes since we are dividing by a negative.

$$\begin{aligned} \frac{22}{-11} &< \frac{\cancel{-11}x}{\cancel{-11}} \\ -2 &< x && \text{(simplify fractions, cancel common factors)} \end{aligned}$$

EXAMPLE Solve $5(x-3) \leq 2(x-3)$.

The goal is still to first isolate a single term with the x in it on one side of the equation. If we multiply or divide by a negative number, we must switch the direction of the inequality.

$$\begin{aligned} \mathbf{5(x-3)} &\leq \mathbf{2(x-3)} \\ \mathbf{5x-15} &\leq \mathbf{2x-6} && \text{(distribution property used in two places)} \\ 5x-\cancel{15}\ \mathbf{\cancel{+15}} &\leq 2x-6\ \mathbf{+15} && \text{(add 15 to both sides (addition principle))} \\ 5x &\leq 2x+9 \\ 5x\ \mathbf{-2x} &\leq \cancel{2x}+9\ \mathbf{\cancel{-2x}} && \text{(subtract } 2x \text{ from both sides)} \\ 3x &\leq 9 \\ \frac{\cancel{3}x}{\mathbf{\cancel{3}}} &\leq \frac{9}{\mathbf{3}} && \text{(divide both sides by 3)} \\ x &\leq 3 && \text{(simplify fraction)} \end{aligned}$$

EXAMPLE Solve $3(2-x) \geq -3x-1$.

$$\begin{aligned} 6-3x &\geq -3x-1 && \text{(distribution property)} \\ 6-3x\ \mathbf{+3x} &\geq -3x-1\ \mathbf{+3x} && \text{(addition property)} \\ 6-\cancel{3x}\ \mathbf{\cancel{+3x}} &\geq -\cancel{3x}-1\ \mathbf{\cancel{+3x}} && \text{(collect like terms)} \\ 6 &\geq -1 \end{aligned}$$

Now we need to interpret what we have found.

Since the result $6 \geq -1$ is true for all x, then any x value makes the original inequality true.

Therefore, the solution is the set $x \in \mathbb{R}$.

EXAMPLE Solve $x \geq x+1$.

$$\begin{aligned} x\ \mathbf{-x} &\geq x+1\ \mathbf{-x} && \text{(subtraction property)} \\ \cancel{x}\ \mathbf{\cancel{-x}} &\geq \cancel{x}+1\ \mathbf{\cancel{-x}} && \text{(cancel like terms)} \\ 0 &\geq 1 \end{aligned}$$

Now we need to interpret what we have found.

Since the result $0 \geq 1$ is false for all x, then no x value makes the original inequality true.

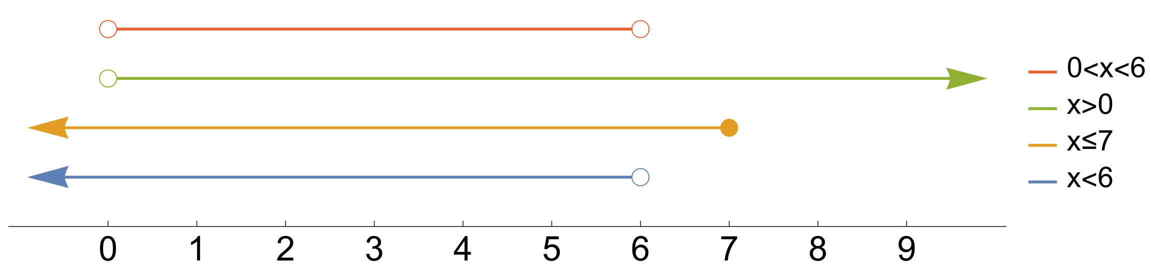
Therefore, the solution is the empty set, which we denote $\{\}$ or $\varnothing$.

EXAMPLE Show the inequalities $3x - 5 < 2x + 1$, $\quad 14 - x \geq x$, $\quad -\frac{1}{4}x < 0$ on a number line. In what interval are all inequalities satisfied?

$$\begin{aligned}
3x - 5 &< 2x + 1 \\
3x - \cancel{5} \mathbf{- 2x + \cancel{5}} &< \cancel{2x} + 1 \mathbf{- \cancel{2x} + 5} && \text{(addition principle)} \\
x &< 6 && \text{(collect like terms)} \\
x &\in (-\infty, 6) && \text{(interval notation)}
\end{aligned}$$

$$\begin{aligned}
14 - x &\geq x \\
\cancel{14} - x \mathbf{- x - \cancel{14}} &\geq \cancel{x} \mathbf{- \cancel{x} - 14} && \text{(addition principle)} \\
-2x &\geq -14 && \text{(collect like terms)} \\
\frac{-2x}{-2} &\leq \frac{-14}{-2} && \text{(division principle; dividing by negative so change direction of inequality)} \\
x &\leq 7 && \text{(simplify)} \\
x &\in (-\infty, 7] && \text{(interval notation)}
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{4}x &< 0 \\
\mathbf{(-4)}\left(-\frac{1}{4}x\right) &> \mathbf{(-4)}(0) && \text{(multiplication principle; dividing by negative so change direction of inequality)} \\
x &> 0 && \text{(simplify)} \\
x &\in (0, \infty) && \text{(interval notation)}
\end{aligned}$$

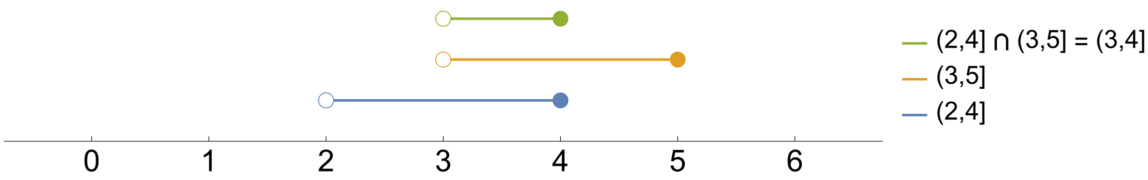


Note all inequalities are satisfied when $x \in (0, 6)$.

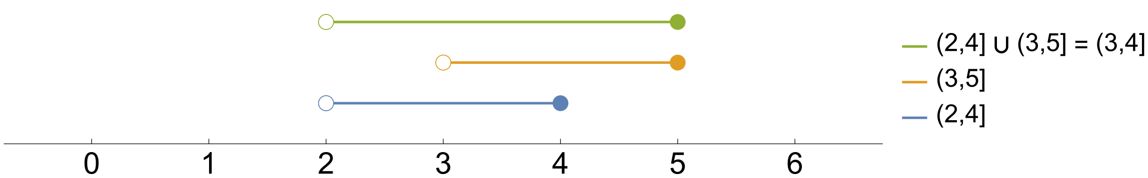
Advice: When trying to interpret inequalities, it is often helpful to draw a number line to help you visualize the inequalities.

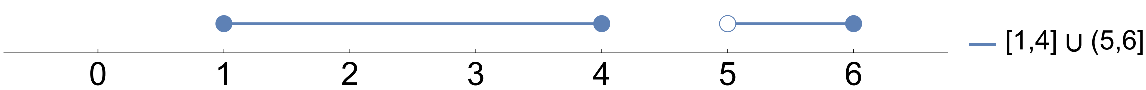
Here are some more advanced interval notations you may see, and will certainly use these often in precalculus.

> **Intersection symbol:** $\cap$ where multiple inequalities are satisfied, for example
>
> $$(2,4] \cap (3,5] = (3,4].$$



> **Union symbol:** $\cup$ where any one of the inequalities are satisfied, for example
>
> $$(2,4] \cup (3,5] = (2,5].$$
>
> $$[1,4] \cup (5,6] \quad \text{(does not simplify).}$$





> **Empty Set:** $\{\}$ or $\varnothing$ where there are no values that satisfy the inequality.
>
> $$[1,4] \cap (5,6] = \{\} \quad \text{(the intervals do not overlap, so the intersection is the empty set).}$$

EXAMPLE Sharon sells sports cars, and can choose between \$100,000 or 8% of her sales as her salary. How much does she need to sell to make the 8% offer the better choice?

Let x be the amount that Sharon sells. We then want 8% of x to be greater than \$100,000.

$$\begin{aligned} 8\%x &> \$100,000 \\ 0.08x &> \$100,000 \\ \frac{\cancel{0.08}x}{\cancel{\mathbf{0.08}}} &> \frac{\$100,000}{\mathbf{0.08}} \\ x &> \$1,250,000 \end{aligned}$$

If Sharon can sell over \$1,250,000 worth of cars, she should take the 8% of her sales as her salary.

3 Solving Word Problems

Word problems appear often in math and science and require us to convert from English to math, then use an appropriate mathematical technique to solve.

A good strategy when you encounter a word problem is to do the following. The strategy is a bit longer than you need right now since it will work in more complicated examples.

1. **Gather Facts.**
   Organize any given information and try to figure out what it is you asked to find.
2. **Draw diagram (optional).**
   If a diagram will be helpful, draw one.
3. **Assign variables.**
   Assign variables to some of the quantities in the problem, especially the unknown quantity you are trying to find.
4. **Write an equation.**
   Here you write down any equations that might help (area of circle, perimeter of a square, etc), and label any quantities with values which are given. Try to find basic formulas or equations between the variables you assigned earlier.
5. **Solve and state the answer.**
   Once you have determined an equation that contains the unknown quantity, solve for the unknown quantity.
6. **Check.**
   If you can, check your answer somehow. Ask yourself if the answer you found is reasonable.

As you can see, for the example above with Sharon selling sports cars, we jumped in at step 3 and assigned x to be the unknown amount that Sharon sells. So the strategy above is not set in stone, it is general advice for solving word problems.

EXAMPLE A Motorcycle shop maintains an inventory of four times as many new bikes as used bikes. If they have space for sixty bikes total, how many new bikes and how many used bikes do they have in stock?

Gather Facts.

There are 60 bikes total.

Assign variables.

There are x used bikes.
There are four times as many new bikes as used bikes, so there are $4x$ new bikes.
The total number of bikes is the sum of the new and used bikes, $4x + x$.

Write an equation.

$$4x + x = 60$$

Solve and state the answer.

$$4x + x = 60 \quad \Rightarrow \quad x = 12.$$

So there are 12 used bikes, and $4 \times 12 = 48$ used bikes in the shop.

Check. $12 + 4(12) = 12 + 48 = 60.$ ✓

The headings are not meant to be overly cumbersome–they are meant to help you organize your thoughts. Use them if they help you, but it is not necessary to use them to create a correct solution.

EXAMPLE A hospital revealed that five officers of the hospital had an average salary of \$125,000 year. Three of the annual salaries are known to be \$50,000, \$60,000, and \$65,000. The annual salaries for the vice president and president were not revealed, but it is known that the president makes twice what the vice president makes. Find the salaries of the president and vice president.

Gather Facts.

The average salary of the five executives is \$125,000.
We know three of the salaries, but don't know two. The ones we know are \$50,000, \$60,000, and \$65,000. The two we don't know are x and $2x$.

Assign variables.
Notice I already did this earlier, now I am just defining what the variables mean!

Let x be the salary of the vice president.
Then $2x$ is the salary of the president.

Write an equation.
The average of five numbers is the sum divided by five.

$$\frac{\$50,000 + \$60,000 + \$65,000 + x + 2x}{5} = \$125,000$$

Solve and state the answer. Solve the equation for x:

$$\frac{\$50,000 + \$60,000 + \$65,000 + x + 2x}{\cancel{5}} \times \cancel{5} = \$125,000 \times \mathbf{5} \qquad \text{(multiplication principle)}$$
$$\cancel{\$175,000} + 3x \cancel{-\$175,000} = \$625,000 - \$175,000 \qquad \text{(addition principle)}$$
$$\frac{\cancel{3}x}{\cancel{3}} = \frac{\$450,000}{\mathbf{3}} \qquad \text{(division principle)}$$
$$x = \$150,000$$

So the vice president earns \$150,000 and the president earns \$300,000.
Check. The average of the salaries should be \$125,000:

$$\frac{\$50,000 + \$60,000 + \$65,000 + \$150,000 + \$300,000}{5} = \frac{\$625,000}{5} = \$125,000\checkmark$$

EXAMPLE In a triangle, the measure of the first angle is twice the measure of the second angle. The measure of the third angle is 20 degrees less than the second angle. What is the measure of each angle?

Let the measure of the second angle be x degrees.
The measure of the first angle is $2x$ degrees.
The measure of the third angle is $x - 20$ degrees.
The sum of the angles must equal 180 degrees: $x + 2x + x - 20 = 180$.

$$4x \cancel{-20} \cancel{+20} = 180 + 20 \qquad \text{(collect like terms, addition principle)}$$
$$\frac{\cancel{4}x}{\cancel{4}} = \frac{200}{\mathbf{4}} \qquad \text{(division principle)}$$
$$x = 50$$

The first angle is 100 degrees, the second angle is 50 degrees, and the third angle is 30 degrees.

Check: $100 + 50 + 30 = 180\checkmark$

EXAMPLE In her statistics course, Jill earned 80/100 and 75/100 on her two chapter tests. The chapter tests count 30% each towards her course grade, and the final exam is worth 40% of her course grade. What must Jill score on the final (out of 100) if she wishes to earn a final grade of at least 83/100?

Gather Facts.

The chapter tests count 30% towards the course grade.
Jill chapter test scores were 80 and 75 out of 100.
The final exam counts 40% towards the course grade.

Assign variables.

Let x be Jill's final exam score, out of 100.

Write an equation.

Jill's course grade (out of 100) will be equal to the following:

$$30\%(80) + 30\%(75) + 40\%(x)$$

To score above 83 in the course, she needs:

$$30\%(80) + 30\%(75) + 40\%(x) > 83$$

Solve and state the answer.

$$\frac{30}{100}(80) + \frac{30}{100}(75) + \frac{40}{100}x > 83 \qquad \text{(convert percents to fractions)}$$

$$\frac{30 \times 80 + 30 \times 75 + 40x}{\cancel{100}} \times \cancel{100} > 83 \times 100 \qquad \text{(multiplication principle)}$$

$$\cancel{4650} + 40x - \cancel{4650} > 8300 - 4650 \qquad \text{(addition principle)}$$

$$\frac{\cancel{40}x}{\cancel{40}} > \frac{3650}{40} \qquad \text{(division principle)}$$

$$x > \frac{365}{4} \sim 91.25$$

Jill needs to score at least 92 out of 100 on the final (assuming she can't earn fractional points) to earn above 83/100 for the course grade.

Check. Assuming she earns 92% on the final, her grade will be

$$30\%(80) + 30\%(75) + 40\%(92) = 83.3 > 83\checkmark$$

Note: If all scores are not out of 100, you can work with fractions for every grade (including the final) to answer problems like this.