1 Solving Systems of linear equations in two unknown variables using algebra

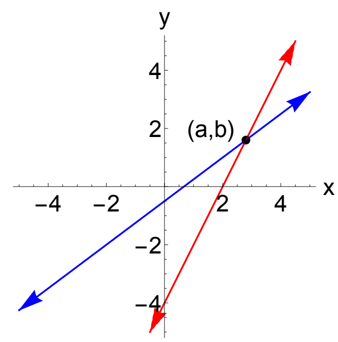
Problems of this type are looking for points of intersection between two lines, and look like:

Solve the system of equations

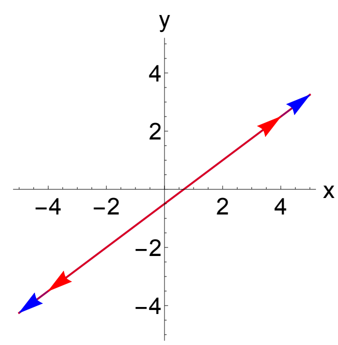
$$3x + 67y = 12,$$
$$45x + 6y = -3.$$

There are three possibilities when algebraically (or graphically) solving two equations in two variables:

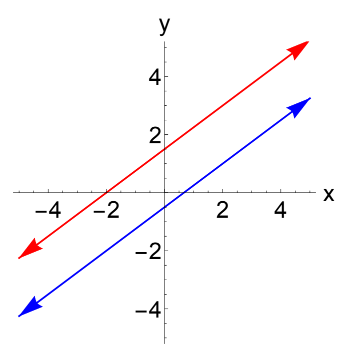
- You will find exactly one ordered pair solution, (a, b).
  - This means the two equations represent two lines which intersect at the **one point** (a, b).
  - We say the equations are **independent** in this case.
  - We say independent equations are **consistent**.



- Algebraically will find something like $12 = 12$ (for example) which is true no matter what the variables are.
  - This means the two equations represent the same line.
  - This means that there is **an infinite number of solution points** where the lines intersect.
  - We say the equations are **dependent** in this case.
  - We say dependent equations are **consistent**.



- Algebraically end up with something like $2 = 5$ (for example) which is never true no matter what the variables are
  - This means the two equations represent two lines which are parallel.
  - This means that there is **no solution**.
  - We say the equations are **inconsistent** in this case.



There are two common methods to algebraically solve a system of equations, and generally an instructor will not care which method you use.

Substitution Method

1. Choose one of the two equations and solve for one variable in terms of the other.
2. Substitute this expression from Step 1 into the *other equation.*
3. You now have one equation in one variable. Solve for the variable.
4. Substitute this value for the variable into one of the original equations to obtain the second variable.
5. Check the solution in both of the original equations.

Elimination Method

1. Arrange each equation in the form $ax + by = c$.
2. Multiply one or both equations by appropriate numbers so that that the coefficients of one of the variables are opposite (for example, -3 and $+3$).
3. Add the two equations from Step 2 so that one variable is eliminated.
4. Solve the resulting equation for the remaining variable.
5. Substitute this value for the variable into one of the original equations to obtain the second variable.
6. Check the solution in both of the original equations.

Barry's Comments

The substitution method can be modified slightly to solve systems of 3 equations with 3 unknown variables (which we won't be studying here in this class), such as

$$\begin{aligned} 23x - 34y + 5z &= 89 \\ 7x - 6y + 22z &= 9 \\ -14x + 4y + z &= 0 \end{aligned}$$

EXAMPLE Solve the system of equations

$$3x + 67y = 12,$$
$$45x + 6y = -3.$$

Solution using substitution method:

Solve the first equation for x:

$$3x + \cancel{67y} \mathbf{-\cancel{67y}} = 12 \mathbf{-67y}$$
$$\frac{\cancel{3}x}{\cancel{\mathbf{3}}} = \frac{12 - 67y}{\mathbf{3}}$$
$$x = \frac{12 - 67y}{3}$$

Substitute into other equation:

$$45\boldsymbol{x} + 6y = -3$$
$$\mathbf{45}\left(\frac{\mathbf{12 - 67}\boldsymbol{y}}{\mathbf{3}}\right) + 6y = -3$$

Solve for y:

$$\mathbf{15 \times \cancel{3}} \times \left(\frac{12 - 67y}{\cancel{3}}\right) + 6y = -3$$
$$15 \times (12 - 67y) + 6y = -3$$
$$15 \times 12 - 15 \times 67y + 6y = -3$$
$$180 - 1005y + 6y = -3$$
$$\cancel{180} - 999y \mathbf{-\cancel{180}} = -3 \mathbf{-180}$$
$$\frac{\cancel{-999}y}{\cancel{\mathbf{-999}}} = \frac{-183}{\mathbf{-999}}$$
$$y = \frac{\cancel{-3} \times 61}{\cancel{-3} \times 333}$$
$$y = \frac{61}{333}$$

Substitute back into any one of the earlier equations:

$$45x + 6\boldsymbol{y} = -3.$$
$$45x + 6\left(\frac{\mathbf{61}}{\mathbf{333}}\right) = -3$$

Solve for x:

$$45x + \cancel{\frac{122}{111}} \mathbf{-\cancel{\frac{122}{111}}} = -3 \mathbf{-\frac{122}{111}}$$
$$45x = -\frac{333}{111} - \frac{122}{111}$$
$$\frac{\cancel{45}x}{\cancel{\mathbf{45}}} = -\frac{455}{\mathbf{45\times}111}$$
$$x = -\frac{\cancel{5} \times 91}{\cancel{5} \times 9 \times 111} = -\frac{91}{999}$$

The solution is $(x, y) = \left(-\frac{91}{999}, \frac{61}{333}\right)$.

Solution using elimination method:

Multiply the first equation by -15 and don't modify the second equation:

$$\mathbf{-15\times}(3x + 67y) = \mathbf{-15\times}12,$$
$$-45x - 1005y = -180$$

Don't modify the second equation, so system is now:

$$\mathbf{-45}\boldsymbol{x}\mathbf{ - 1005}\boldsymbol{y}\mathbf{ = -180}$$
$$\mathbf{45}\boldsymbol{x}\mathbf{ + 6}\boldsymbol{y}\mathbf{ = -3}$$

Add the two equations (blue is first equation, red second):

$$\cancel{\mathbf{-45}\boldsymbol{x}} \mathbf{- 1005}\boldsymbol{y} + \cancel{\mathbf{45}\boldsymbol{x}} \mathbf{+ 6}\boldsymbol{y} = \mathbf{-180} + \mathbf{(-3)}$$
$$-999y = -183$$

Solve for y:

$$\frac{\cancel{-999}y}{\cancel{\mathbf{-999}}} = \frac{-183}{\mathbf{-999}}$$
$$y = \frac{61}{333}$$

Substitute back into any one of the earlier equations:

$$45x + 6\boldsymbol{y} = -3.$$
$$45x + 6\left(\frac{\mathbf{61}}{\mathbf{333}}\right) = -3$$

Solve for x:

$$45x + \cancel{\frac{122}{111}} \mathbf{-\cancel{\frac{122}{111}}} = -3 \mathbf{-\frac{122}{111}}$$
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$$x = -\frac{\cancel{5} \times 91}{\cancel{5} \times 9 \times 111} = -\frac{91}{999}$$

The solution is $(x, y) = \left(-\frac{91}{999}, \frac{61}{333}\right)$.

EXAMPLE Solve the system of equations

$$\begin{aligned} 2(y-3) &= x+3y, \\ x+2 &= 3-y. \end{aligned}$$

First, write both equations in the form $ax+by=c$. This isn't strictly necessary, but I find it helpful.

$$\begin{aligned} \text{First Equation: } 2(y-3) &= x+3y && \text{(distribute 2)} \\ 2y - \cancel{6} \mathbf{-x-3y+\cancel{6}} &= \cancel{x} + \cancel{3y} \mathbf{-\cancel{x} - \cancel{3y} + 6} && \text{(addition property)} \\ -x-y &= 6 && \text{(combine like terms)} \end{aligned}$$

$$\begin{aligned} \text{Second Equation: } x + \cancel{2} \mathbf{+y-\cancel{2}} &= 3 - \cancel{y} \mathbf{+\cancel{y} - 2} && \text{(addition property)} \\ x+y &= 1 && \text{(combine like terms)} \end{aligned}$$

The system of equations is now:

$$\begin{aligned} \boldsymbol{-x-y} &= \mathbf{6} \\ \boldsymbol{x+y} &= \mathbf{1} \end{aligned}$$

Solve with the elimination method. Add the two equations.

$$\begin{aligned} \cancel{\boldsymbol{-x-y}} + \cancel{\boldsymbol{x+y}} &= \mathbf{6} + \mathbf{1} \\ 0 &= 7 && \text{(false)} \end{aligned}$$

This is never true. The system of equations has no solution since it is an inconsistent system of equations.

EXAMPLE Solve the system of equations

$$\begin{aligned} 21x - 7y &= 14, \\ -21x + 7y &= -14. \end{aligned}$$

Solve the first equation for x:

$$\begin{aligned} 21x - \cancel{7y} \mathbf{+\cancel{7y}} &= 14 \mathbf{+7y} && \text{(addition principle)} \\ \frac{\cancel{21}x}{\cancel{\mathbf{21}}} &= \frac{14+7y}{\mathbf{21}} && \text{(division principle)} \\ x &= \frac{14+7y}{21} \end{aligned}$$

Substitute into the second equation, and then solve for y:

$$\begin{aligned} -21\boldsymbol{x} + 7y &= -14 \\ -21\left(\frac{\mathbf{14+7y}}{\mathbf{21}}\right) + 7y &= -14 && \text{(note the minus sign with the 21 does not cancel!)} \\ \cancel{21}\left(\frac{-14-7y}{\cancel{21}}\right) + 7y &= -14 && \text{(distribute the minus sign, then cancel the 21)} \\ -14 - \cancel{7y} + \cancel{7y} &= -14 \\ -14 &= -14 && \text{(true for all } x\text{)} \end{aligned}$$

This is true no matter what x is. There are an infinite number of ordered pairs that satisfy the system of equations.

If we want to express the infinite number of ordered pairs algebraically, here is one way to do it: The two equations in the system are actually *equivalent equations*, so we can pick either one of them as the representation of the infinite number of ordered pairs. Let's pick the first, $21x - 7y = 14$. This equation ($21x - 7y = 14$) represents all the ordered pairs that satisfy the system of equations.

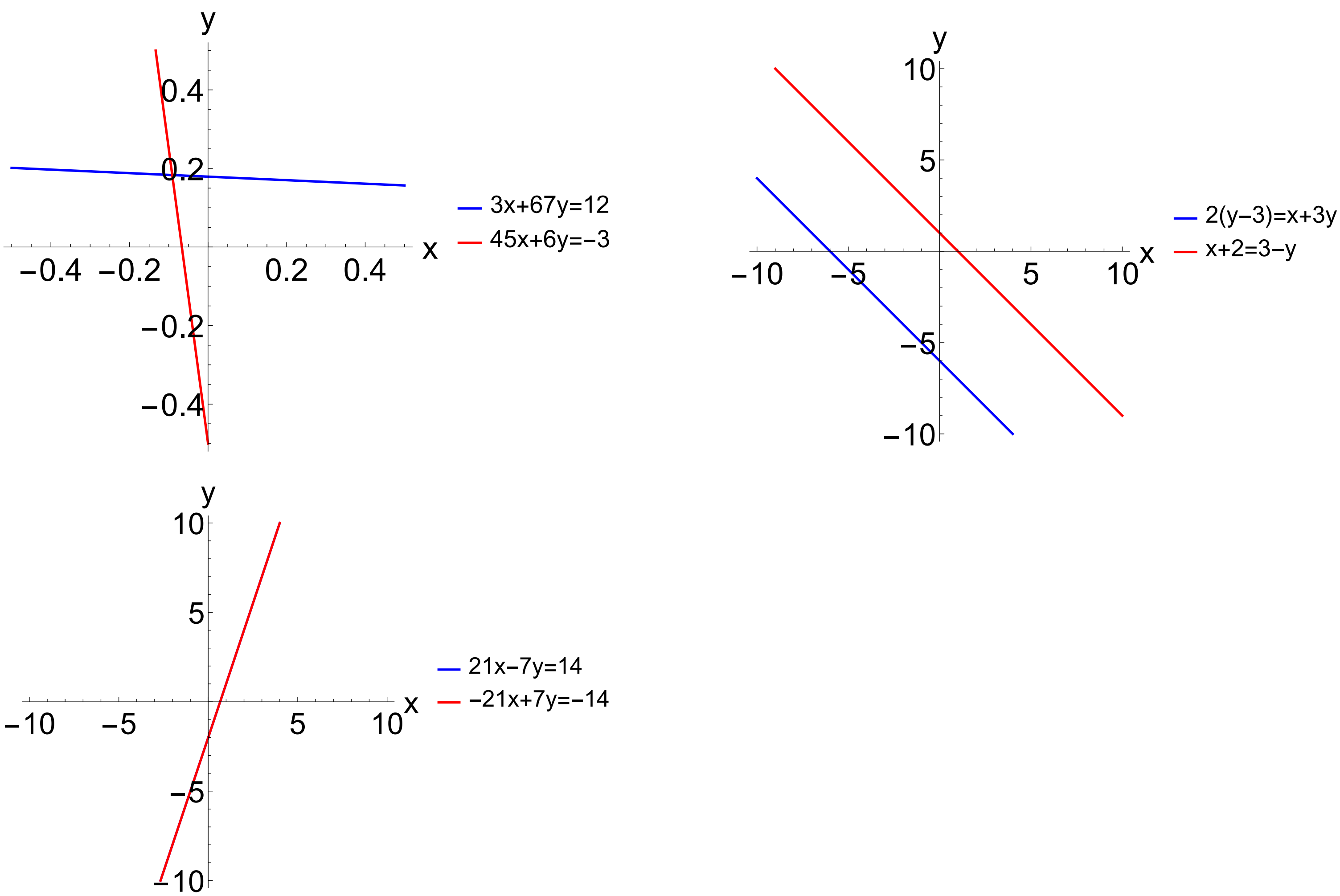
If we want to make it a bit clearer that this represents an infinite set of ordered pairs, we can solve for y:

$$\begin{aligned} 21x - 7y &= 14 \\ -7y &= 14 - 21x \\ y &= -2 + 3x \end{aligned}$$

and we can express the set of ordered pairs as $(x, y) = (x, -2 + 3x)$.

Aside: What we have done in this last step is create a *parametric relation*, since we could also write the ordered pairs as $(t, -2 + 3t)$ and we get different ordered pairs by picking different values of the parameter t. You will see parametric relations in precalculus.

Here are plots of the equations in the previous three examples.



EXAMPLE The Tupper Farm has 450 acres of land allotted for raising corn and wheat. The cost to cultivate corn in \$42 per acre. The cost to cultivate wheat is \$35 per acre. The Tuppers have \$16,520 available to cultivate these crops. How many acres of each crop should the Tuppers plant?

Assign variables.

x is the number of acres of corn they plant.
y is the number of acres of wheat they plant.

Gather Facts.

450 acres of land allotted for raising corn and wheat.
The cost to cultivate corn in \$42 per acre.
The cost to cultivate wheat is \$35 per acre.
The Tuppers have \$16,520 available to cultivate these crops.

Write equations.

$$\begin{aligned} x + y &= 450. && \text{(comes from how many acres they have)} \\ \$42x + \$35y &= \$16,520. && \text{(comes from how much money they have)} \end{aligned}$$

Solve and state the answer. The two equations we need to solve are

$$\begin{aligned} x + y &= 450 \\ 42x + 35y &= 16,520 \end{aligned}$$

Solve with the elimination method. multiply the first equation by -35:

$$\begin{aligned} \mathbf{-35x - 35y} &= \mathbf{-15,750} \\ \mathbf{42x + 35y} &= \mathbf{16,520} \end{aligned}$$

Add the equations

$$\begin{aligned} \mathbf{-35x - \cancel{35y}} + \mathbf{42x + \cancel{35y}} &= \mathbf{-15,750} + \mathbf{16,520} \\ \frac{\cancel{7}x}{\cancel{7}} &= \frac{770}{\mathbf{7}} && \text{(combine like terms, division principle)} \\ x &= 110. \end{aligned}$$

Substitute back into an earlier equation and solve for y:

$$\begin{aligned} \mathbf{x} + y &= 450 \\ \mathbf{110} + y &= 450 \\ \cancel{110} + y \mathbf{-\cancel{110}} &= 450 \mathbf{-110} && \text{(addition principle)} \\ y &= 340 \end{aligned}$$

The Tuppers can plant 110 acres of corn and 340 acres of wheat.

2 Solving Systems of Inequalities by Sketching

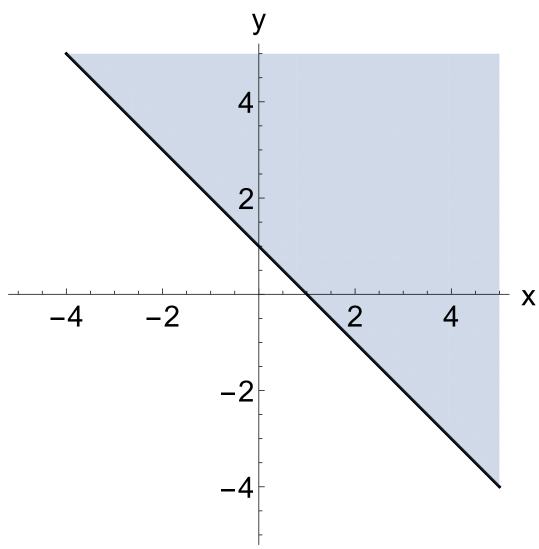
The technique to do this just combines the techniques for graphing one inequality in two variables and finding the point of intersection of two lines.

> **Feasible Region:** The region where all the inequalities are satisfied. The feasible region represents all the ordered pairs which satisfy all the inequalities.

EXAMPLE Sketch the feasible region for the following inequalities:

$$x + y \geq 1,$$
$$x - 3y \geq 2,$$
$$x > 2.$$

Sketch $x + y \geq 1$:



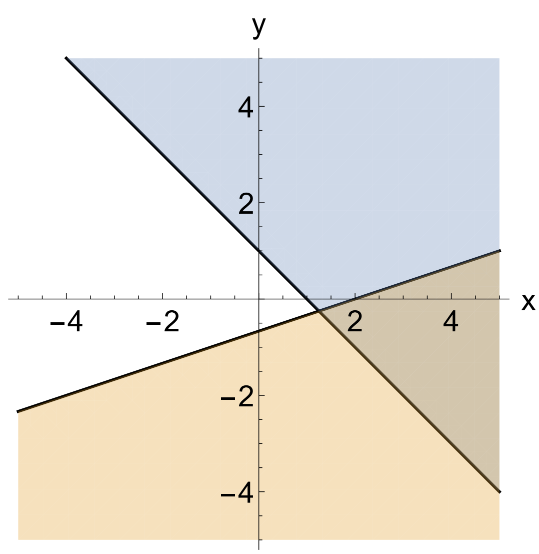
Since line satisfies inequality, sketch solid line $x + y = 1$.

When $x = 0$, then $0 + y = 1$, so $y = 1$.
Point on line is $(0, 1)$.

When $y = 0$, then $x + 0 = 1$, so $x = 1$.
Point on line is $(1, 0)$.

Test point $(0, 0)$: $0 + 0 \geq 1$ is False,
so shade side opposite $(0, 0)$.

Sketch $x - 3y \geq 2$:



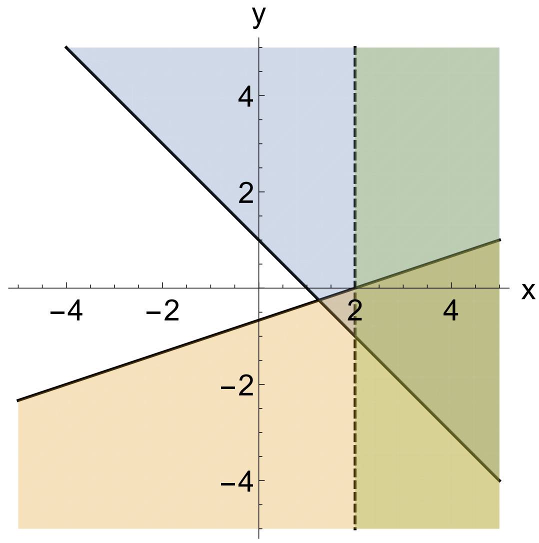
Since line satisfies inequality, sketch solid line $x - 3y = 2$.

When $x = 0$, then $0 - 3y = 1$, so $y = -1/3$.
Point on line is $(0, -1/3)$.

When $y = 0$, then $x + 0 = 2$, so $x = 2$.
Point on line is $(2, 0)$.

Test point $(0, 0)$: $0 + 0 \geq 2$ is False,
so shade side opposite $(0, 0)$.

Sketch $x > 2$**:**



Since line does not satisfy inequality, sketch dashed line $x = 2$.

This is just a vertical line.

Test point $(0, 0)$: $0 > 2$ is False,
so shade side opposite $(0, 0)$.

Here is the region that satisfies all three inequalities identified (the feasible region):