1 Exponents

Rules of Exponents:

1. $x^0 = 1$ if $x \neq 0$ (0^0 is indeterminant and is dealt with in calculus).
2. Product Rule: $x^a \times x^b = x^{a+b}$.
3. Quotient Rule: $\frac{x^a}{x^b} = x^{a-b}$.
4. Power Rule: $(x^a)^b = x^{ab}$.
5. Product Raised to Power Rule: $(xy)^a = x^a y^a$.
6. Quotient Raised to a Power Rule: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$ if $y \neq 0$.
7. Negative Exponent: $x^{-n} = \frac{1}{x^n}$, if $x \neq 0$.

The **rules of exponents** are tremendously important, and it is critical that you memorize them. They will be useful in a variety of ways for the rest of the course and beyond.

Notice I did not break the quotient rule up into a bunch of cases–that isn't necessary if you are comfortable working with negative exponents (which you will be!).

EXAMPLE Simplify by combining exponents in $\left(\frac{3x^{-3}y^{-2}}{x^{-2}y^{-4}}\right)^2$. Make all exponents positive in your final answer.

$$\begin{aligned}
\left(\frac{3x^{-3}y^{-2}}{x^{-2}y^{-4}}\right)^{\mathbf{2}} &= \frac{3^{\mathbf{2}}(x^{-3})^{\mathbf{2}}(y^{-2})^{\mathbf{2}}}{(x^{-2})^{\mathbf{2}}(y^{-4})^{\mathbf{2}}} && \text{(product/quotient raised to power rules)} \\
&= \frac{9(\boldsymbol{x^{-3}})^{\mathbf{2}}(\boldsymbol{y^{-2}})^{\mathbf{2}}}{(\boldsymbol{x^{-2}})^{\mathbf{2}}(y^{-4})^2} && \text{(no changes, highlighting power rules)} \\
&= \frac{9\boldsymbol{x^{-6}y^{-4}}}{\boldsymbol{x^{-4}}y^{-8}} \\
&= \frac{9\boldsymbol{x^{-6}y^{-4}}}{\boldsymbol{x^{-4}y^{-8}}} && \text{(no changes, highlighting quotient rules)} \\
&= 9\boldsymbol{x^{-6+4}y^{-4+8}} \\
&= 9\boldsymbol{x^{-2}}y^4 && \text{(negative exponent rule)} \\
&= \frac{9y^4}{\boldsymbol{x^2}}
\end{aligned}$$

You can use the rules in different orders (as long as you are careful) and you will arrive at the same result.

2 Polynomials

- A **polynomial** is the sum of a finite number of terms of the form ax^n where a is any real number and n is a whole number.
- A multivariable polynomial has more than one polynomial, for example $56x^3y^4 + xy^2 - x^2$.
- The **degree of a term** is the sum of the exponents of all the variables in the term.
- The **degree of a polynomial** is the largest degree of all the terms in the polynomial.
- **monomials** are polynomials with one term.
- **binomials** are polynomials with two terms.
- **trinomials** are polynomials with three terms.

2.1 Addition of Polynomials

Add or subtract two polynomials by collecting like terms.

EXAMPLE Simplify $\left(3x^2 - \frac{7}{30}x + 2\right) - \left(\frac{7}{99}x^4 - \frac{1}{6}x + 2\right)$.

$$\begin{aligned}
\left(3x^2 - \frac{7}{30}x + 2\right) - \left(\frac{7}{99}x^4 - \frac{1}{6}x + 2\right) &= 3x^2 - \frac{7}{30}x + 2 - \frac{7}{99}x^4 + \frac{1}{6}x - 2 && \text{(distribute)} \\
&= 3x^2 - \frac{7}{30}x \cancel{+ 2} - \frac{7}{99}x^4 + \frac{1}{6}x \cancel{- 2} && \text{(identify like terms)} \\
&= -\frac{7}{99}x^4 + 3x^2 + \left(-\frac{7}{30} + \frac{1}{6}\right)x && \text{(collect like terms)} \\
&= -\frac{7}{99}x^4 + 3x^2 + \left(-\frac{7}{30} + \frac{1\times 5}{6\times 5}\right)x && \text{(add fraction)} \\
&= -\frac{7}{99}x^4 + 3x^2 + \left(-\frac{7}{30} + \frac{5}{30}\right)x \\
&= -\frac{7}{99}x^4 + 3x^2 + \left(\frac{-7+5}{30}\right)x \\
&= -\frac{7}{99}x^4 + 3x^2 - \frac{1}{15}x
\end{aligned}$$

2.2 Multiplication of Polynomials

Multiply polynomials by using the distributive property, then collect like terms.

EXAMPLE Simplify $(x + 2x^2 + 4x^3)(1 + x)$.

$$\begin{aligned}
(x + 2x^2 + 4x^3)(1 + x) &= x(1 + x) + 2x^2(1 + x) + 4x^3(1 + x) && \text{(distribute)} \\
&= x(1 + x) + 2x^2(1 + x) + 4x^3(1 + x) && \text{(no changes, just highlight)} \\
&= x + x^2 + 2x^2 + 2x^3 + 4x^3 + 4x^4 && \text{(distribute)} \\
&= x + x^2 + 2x^2 + 2x^3 + 4x^3 + 4x^4 && \text{(no changes, highlight like terms)} \\
&= x + 3x^2 + 6x^3 + 4x^4 && \text{(collect like terms)}
\end{aligned}$$

Special cases of multiplication

These occur frequently, so they are useful to know, but you could always work these out using the distributive property.

$$\begin{aligned} (a+b)(a-b) &= a^2 - b^2 && \text{(difference of squares)} \\ (a+b)^2 &= a^2 + 2ab + b^2 && \text{(binomial with addition)} \\ (a-b)^2 &= a^2 - 2ab + b^2 && \text{(binomial with subtraction)} \end{aligned}$$

Here are where these come from, using the distributive property:

$$\begin{aligned} \boldsymbol{(a+b)}(a-b) &= \boldsymbol{(a+b)}a - \boldsymbol{(a+b)}b && \text{(distribute)} \\ &= (a+b)\boldsymbol{a} - (a+b)\boldsymbol{b} && \text{(highlight next distributions)} \\ &= \boldsymbol{a^2} + \boldsymbol{ba} - (\boldsymbol{ab} + \boldsymbol{b^2}) && \text{(distribute)} \\ &= a^2 \cancel{+ ba - ab} - b^2 && \text{(distribute minus sign, note } ab = ba\text{)} \\ &= a^2 - b^2 \end{aligned}$$

$$\begin{aligned} (a+b)^2 = \boldsymbol{(a+b)}(a+b) &= \boldsymbol{(a+b)}a + \boldsymbol{(a+b)}b && \text{(distribute)} \\ &= (a+b)\boldsymbol{a} + (a+b)\boldsymbol{b} && \text{(highlight next distributions)} \\ &= \boldsymbol{a^2} + \boldsymbol{ba} + \boldsymbol{ab} + \boldsymbol{b^2} && \text{(distribute)} \\ &= a^2 \boldsymbol{+ ba + ab} + b^2 && \text{(collect like terms, note } ab = ba\text{)} \\ &= a^2 + 2ab + b^2 \end{aligned}$$

$$\begin{aligned} (a-b)^2 = \boldsymbol{(a-b)}(a-b) &= \boldsymbol{(a-b)}a - \boldsymbol{(a-b)}b && \text{(distribute)} \\ &= (a-b)\boldsymbol{a} - (a-b)\boldsymbol{b} && \text{(highlight next distributions)} \\ &= \boldsymbol{a^2} - \boldsymbol{ba} - (\boldsymbol{ab} - \boldsymbol{b^2}) && \text{(distribute)} \\ &= a^2 - ba - ab + b^2 && \text{(distribute minus sign)} \\ &= a^2 \boldsymbol{- ba - ab} + b^2 && \text{(collect like terms, note } ab = ba\text{)} \\ &= a^2 - 2ab + b^2 \end{aligned}$$

Aside: You may have seen vertical multiplication of polynomials. Use that if you like, it is mathematically correct. But I far prefer using the distributive property to multiply polynomials, and I think you will too with a bit of practice. Add as many intermediate steps as you need to get the correct final answer!

EXAMPLE Multiply $(3ab)(5a^2c)(-2b^2c^3)$.

$$\begin{aligned} (3ab)(5a^2c)(-2b^2c^3) &= -(3ab)(5a^2c)(2b^2c^3) && \text{(deal with the overall sign first)} \\ &= -(3 \cdot 5 \cdot 2)(aa^2)(bb^2)(cc^3) && \text{(collect together constants and factors with the same base)} \\ &= -(30)(a^3)(b^3)(c^4) && \text{(simplify using rules of exponents, remember } a = a^1\text{)} \\ &= -30a^3b^3c^4 && \text{(remove extraneous parentheses)} \end{aligned}$$

EXAMPLE Multiply $(9a - 14b)^2$.

$$\begin{aligned}
(9a-14b)^2 &= (9a-14b)\mathbf{(9a-14b)} && \text{(expand the exponent)}\\
&= \mathbf{9a(9a-14b)-14b(9a-14b)} && \text{(distributed the } 9a-14b \text{ factor)}\\
&= \mathbf{9a}(9a) - \mathbf{9a}(14b) + \mathbf{(-14b)}9a - \mathbf{(-14b)}14b && \text{(distribute the } 9a \text{ factor and the } -14b \text{ factors)}\\
&= 81a^2 - 126ab - 126ab + 196b^2 && \text{(multiply)}\\
&= 81a^2 - 256ab + 196b^2
\end{aligned}$$

EXAMPLE Multiply $(x^2 - 5x)(x^5 + 7x)(x^3 + x + 3)$.

This is a test of your ability to keep tracks of all the pieces during a long solution.

Multiply the first two factors together:

$$\begin{aligned}
\mathbf{(x^2-5x)}(x^5+7x) &= \mathbf{(x^2-5x)(x^5)} + \mathbf{(x^2-5x)(7x)} && \text{(distribute the } x^2-5x \text{ factor)}\\
&= x^2\mathbf{(x^5)} - 5x\mathbf{(x^5)} + x^2\mathbf{(7x)} - 5x\mathbf{(7x)} && \text{(distribute the factors into the } x^2-5x\text{)}\\
&= x^{2+5} - 5x^{1+5} + 7x^{2+1} - 35x^{1+1} && \text{(simplify using exponent rule } x^a \cdot x^b = x^{a+b}\text{)}\\
&= x^7 - 5x^6 + 7x^3 - 35x^2
\end{aligned}$$

Now for the final polynomial product:

$(x^2 - 5x)(x^5 + 7x)(x^3 + x + 3)$

$= (x^7 - 5x^6 + 7x^3 - 35x^2)\mathbf{(x^3 + x + 3)}$

(distribute the $x^3 + x + 3$ factor)

$= x^7\mathbf{(x^3 + x + 3)} - 5x^6\mathbf{(x^3 + x + 3)} + 7x^3\mathbf{(x^3 + x + 3)} - 35x^2\mathbf{(x^3 + x + 3)}$

(distribute the factors into the $x^3 + x + 3$)

$= (x^7)x^3 + (x^7)x + (x^7)3 + (-5x^6)x^3 + (-5x^6)x + (-5x^6)3 + (7x^3)x^3$
$+ (7x^3)x + (7x^3)3 + (-35x^2)x^3 + (-35x^2)x + (-35x^2)3$

(simplify using exponent rule $x^a \cdot x^b = x^{a+b}$)

$= x^{7+3} + x^{7+1} + 3x^7 - 5x^{6+3} - 5x^{6+1} - 15x^6 + 7x^{3+3} + 7x^{3+1} + 21x^3 - 35x^{2+3} - 35x^{2+1} - 105x^2$

$= x^{10} + x^8 + \mathbf{3x^7} - 5x^9 - \mathbf{5x^7} - \mathbf{15x^6} + \mathbf{7x^6} + 7x^4 + \mathbf{21x^3} - 35x^5 - \mathbf{35x^3} - 105x^2$

(collect like terms)

$= x^{10} - 5x^9 + x^8 - \mathbf{2x^7} - \mathbf{8x^6} - 35x^5 + 7x^4 - \mathbf{14x^3} - 105x^2$

It is important to be good at multiplication of polynomials, since soon we will look at factoring of polynomials, which is in essence the opposite of multiplication of polynomials (and yes, factoring is related to polynomial division).

The order you distribute in when multiplying polynomials does not matter.

$$\begin{aligned}
(x^2-4x+5)(x-2) &= (x^2-4x+5)x + (x^2-4x+5)(-2) && \text{(distribute the } x^2-4x+5 \text{ term)}\\
&= x^3-4x^2+5x+x^2(-2)-4x(-2)+5(-2) && \text{(distribute the } x \text{ and } -2)\\
&= x^3-4x^2+5x-2x^2+8x-10 && \text{(multiply)}\\
&= x^3-4x^2+5x-2x^2+8x-10 && \text{(collect like terms)}\\
&= x^3-6x^2+13x-10
\end{aligned}$$

Here's an alternate simplification:

$$\begin{aligned}
(x^2-4x+5)(x-2) &= x^2(x-2)-4x(x-2)+5(x-2) && \text{(distribute the } x-2 \text{ term)}\\
&= x^3-2x^2-4x^2+8x+5x-10 && \text{(now distribute the } x^2\text{, } -4x\text{, and 5)}\\
&= x^3-2x^2-4x^2+8x+5x-10 && \text{(collect like terms)}\\
&= x^3-6x^2+13x-10
\end{aligned}$$

Note: FOIL (First, Outside, Inside, Last) obscures the distributive property you are using, and only works on binomials, so I don't use it. The distributive property is what I use. You are, as always, free to use it since FOIL is mathematically correct.

2.3 Division of Polynomials

When you have a ratio of polynomials (called a **rational expression**, which we will look at in more detail later) where the degree of the polynomial in the numerator is larger than the degree of the polynomial in the denominator requires us to divide polynomials. For example, we need a way to simplify

$$(15x^2+4x+6) \div (5x-2) \qquad \text{or} \qquad \frac{15x^2+4x+6}{5x-2}.$$

Dividing polynomials is best done by the process of polynomial long division.

The technique for polynomial long division is very similar to the technique for dividing an improper fraction.

EXAMPLE Write the improper fractions as mixed numbers: (a) $\frac{100}{6}$ (b) $\frac{8390}{5}$ (c) $\frac{9999}{8}$

$$\begin{array}{r} 16 \\ 6\,\overline{)\,100} \\ \underline{60} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

So $\frac{100}{6} = 16 + \frac{4}{6} = 16\frac{2}{3}$

$$\begin{array}{r} 1678 \\ 5\,\overline{)\,8390} \\ \underline{5000} \\ 3390 \\ \underline{3000} \\ 390 \\ \underline{350} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

So $\frac{8390}{5} = 1678 + \frac{0}{6} = 1678$

$$\begin{array}{r} 1249 \\ 8\,\overline{)\,9999} \\ \underline{8000} \\ 1999 \\ \underline{1600} \\ 399 \\ \underline{320} \\ 79 \\ \underline{72} \\ 7 \end{array}$$

So $\frac{9999}{8} = 1249 + \frac{7}{8} = 1249\frac{7}{8}$

In the first example, 6 is the **divisor**, 100 is the **dividend**, 16 is the **quotient**, and 4 is the **remainder**.

For long division, you divide one integer into another by:

1. Align columns that represent powers of 10.
2. Work from left to right (largest power of 10 to smallest), looking for how many times the divisor "goes into" another number, which creates the quotient from left to right.
3. Multiply the number placed in the quotient by the divisor, and subtract.
4. Repeat the process until the remainder is less than the divisor.

To modify this to work for polynomials, you need to:

1. Align columns of power of x (making sure to include a column for any missing powers).
2. Work from left to right (largest power of x to smallest), looking for how many times the leading term in the divisor "goes into" the leading term on the dividend.
   You work with the leading terms to determine the terms in the quotient, and the trailing terms just follow along through the calculation.
3. Multiply the term placed in the quotient by the divisor, and subtract.
   This **subtraction** is usually performed by **multiplying by -1 and adding** (this is done to ensure the signs of the trailing terms are correct).
4. Repeat the process until the degree of the remainder is less than the degree of the divisor.

Determining how many times leading terms "goes into" another number
Here is how to compute this when the division is not even or there are minus signs.
EXAMPLE How many times does $4x$ go into $-5x^2$?
This is the same as asking what we have to multiply $4x$ by to get $-5x^2$.

$$4x \times w = -5x^2$$
$$w = -\frac{5x^2}{4x} = -\frac{5}{4}x$$

So $4x$ goes into $-5x^2$ a total of $-\frac{5}{4}x$ times.

EXAMPLE Do polynomial long division for $15x^2 + 4x + 6$ divided by $5x - 2$.

$$\begin{array}{r|rrr} & & 3x & +2 \\ \hline 5x-2 & 15x^2 & +4x & +6 \\ & -15x^2 & +6x & \\ \hline & & 10x & +6 \\ & & -10x & +4 \\ \hline & & & 10 \end{array}$$

1. Leading terms: since $5x \times 3x = 15x^2$, $5x$ goes into $15x^2$ a total of $3x$ times.
2. Multiply $3x \times (5x - 2) = 15x^2 - 6x$, **which is subtracted by adding $-15x^2 + 6x$** which leaves $10x + 6$.
3. Repeat. Leading terms, since $5x \times 2 = 10x$, $5x$ goes into $10x$ a total of 2 times.
4. Multiply $2 \times (5x - 2) = 10x - 4$, **which is subtracted by adding $-10x + 4$** which leaves 10.
5. The remainder is 10.

This computation shows that $\dfrac{15x^2 + 4x + 6}{5x - 2} = 3x + 2 + \dfrac{10}{3x + 2}$.

EXAMPLE Do polynomial long division for $\dfrac{20x^2 - 17x + 3}{4x - 1}$

$$
\begin{array}{r|l}
 & \quad 5x - 3 \\
\hline
4x - 1 & \; 20x^2 - 17x + 3 \\
 & -20x^2 + 5x \\
\hline
 & \quad -12x + 3 \\
 & \quad\;\; 12x - 3 \\
\hline
 & \qquad\quad 0
\end{array}
$$

Reading this:

1. Leading terms: since $4x \times 5x = 20x^2$, $4x$ goes into $20x^2$ a total of $5x$ times.
2. Multiply $5x \times (4x - 1) = 20x^2 - 5x$, **which is subtracted by adding $-20x^2 + 5x$** which leaves $-12x + 3$.
3. Repeat. Leading terms, since $4x \times (-3) = -12x$, $4x$ goes into $-12x$ a total of -3 times.
4. Multiply $-3 \times (4x - 1) = -12x + 3$, **which is subtracted by adding $12x - 3$** which leaves 0.
5. The remainder is 0, so the division was even.

This computation shows that $\dfrac{20x^2 - 17x + 3}{4x - 1} = 5x - 3 + \dfrac{0}{4x - 1} = 5x - 3, \quad 4x - 1 \neq 0.$

We include the $4x - 1 \neq 0$ to include the fact that the original expression was not defined when $4x - 1 = 0$ due to a division by zero.

EXAMPLE Do polynomial long division for $\dfrac{6x^3 - x^2 + 3x + 10}{2x - 1}$

$$
\begin{array}{r|l}
 & \qquad 3x^2 + x + 2 \\
\hline
2x - 1 & \; 6x^3 - x^2 + 3x + 10 \\
 & -6x^3 + 3x^2 \\
\hline
 & \qquad 2x^2 + 3x \\
 & \quad -2x^2 + x \\
\hline
 & \qquad\qquad 4x + 10 \\
 & \qquad\quad -4x + 2 \\
\hline
 & \qquad\qquad\quad 12
\end{array}
$$

Reading this:

1. Leading terms: since $2x \times 3x^2 = 6x^3$, $2x$ goes into $6x^3$ a total of $3x^2$ times.
2. Multiply $3x^2 \times (2x - 1) = 6x^3 - 3x^2$, **which is subtracted by adding $-6x^3 + 3x^2$** which leaves $2x^2$.
3. Repeat. Leading terms, since $2x \times x = 2x^2$, $2x$ goes into $2x^2$ a total of x times.
4. Multiply $x \times (2x - 1) = 2x^2 - x$,**which is subtracted by adding $-2x^2 + x$** which leaves $4x$.
5. Repeat. Leading terms, since $2x \times 2 = 4x$, $2x$ goes into $4x$ a total of 2 times.
6. Multiply $2 \times (2x - 1) = 4x - 2$, **which is subtracted by adding $-4x + 2$** which leaves 12.
7. The remainder is 12.

This computation shows that $\dfrac{6x^3 - x^2 + 3x + 10}{2x - 1} = 3x^2 + x + 2 + \dfrac{12}{2x - 1}$.

EXAMPLE Do polynomial long division for $\dfrac{6x^3 - 3x^2 + 4}{2x + 1}$

Notice the numerator does not have an x^1 term, but we need a column for that when we do the long division!

$$
\begin{array}{rr}
 & 3x^2 - 3x + \frac{3}{2} \\
2x + 1 \big) & \overline{6x^3 - 3x^2 \qquad + 4} \\
 & \underline{-\,6x^3 - 3x^2 \qquad\quad} \\
 & -\,6x^2 \qquad\quad \\
 & \underline{6x^2 + 3x \quad} \\
 & 3x + 4 \\
 & \underline{-\,3x - \frac{3}{2}} \\
 & \frac{5}{2}
\end{array}
$$

Reading this:

1. Leading terms: since $2x \times 3x^2 = 6x^3$, $2x$ goes into $6x^3$ a total of $3x^2$ times.
2. Multiply $3x^2 \times (2x + 1) = 6x^3 + 3x^2$, **which is subtracted by adding $-6x^3 - 3x^2$** which leaves $-6x^2$.
3. Repeat. Leading terms, since $2x \times (-3x) = -6x^2$, $2x$ goes into $-6x^2$ a total of $-3x$ times.
4. Multiply $-3x \times (2x + 1) = -6x^2 - 3x$, **which is subtracted by adding $6x^2 + 3x$** which leaves $3x$.
5. Repeat. Leading terms, since $2x \times \frac{3}{2} = 3x$, $2x$ goes into $3x$ a total of $\frac{3}{2}$ times.
6. Multiply $\frac{3}{2} \times (2x + 1) = 3x + \frac{3}{2}$, **which is subtracted by adding $-3x - \frac{3}{2}$** which leaves $\frac{5}{2}$.
7. The remainder is $\frac{5}{2}$.

This computation shows that $\dfrac{6x^3 - 3x^2 + 4}{2x + 1} = 3x^2 - 3x + \dfrac{3}{2} + \dfrac{5/2}{2x + 1}$.

We can check if this is correct by finding a common denominator on the right hand side, which will involve polynomial multiplication:

$$
\begin{aligned}
3x^2 - 3x + \frac{3}{2} + \frac{5/2}{2x+1} &= \left(3x^2 - 3x + \frac{3}{2}\right)\frac{\mathbf{2x+1}}{\mathbf{2x+1}} + \frac{\frac{5}{2}}{2x+1} && \text{(common denominator)} \\
&= \frac{\left(3x^2 - 3x + \frac{3}{2}\right)\mathbf{(2x+1)} + \frac{5}{2}}{2x+1} && \text{(add fractions)} \\
&= \frac{3x^2\mathbf{(2x+1)} - 3x\mathbf{(2x+1)} + \frac{3}{2}\mathbf{(2y+1)} + \frac{5}{2}}{2y+1} && \text{(distribute)} \\
&= \frac{3x^2(2x) + 3x^2(1) + (-3x)2x + (-3x)1 + \frac{3}{2}(2x) + \frac{3}{2} + \frac{5}{2}}{2x+1} && \text{(distribute)} \\
&= \frac{6x^3 + 3x^2 - 6x^2 - \cancel{3x} + \cancel{3x} + \frac{3+5}{2}}{2x+1} && \text{(multiply)} \\
&= \frac{6x^3 - 3x^2 + 4}{2x+1}
\end{aligned}
$$

which verifies the long division result. Obviously, checking your long division in this manner gives you good practice with polynomial multiplication as well!

EXAMPLE Do polynomial long division for $\dfrac{4x^3 - x^2 - 4x + 1}{x^2 - 1}$

Note that in this example, we are not dividing by a linear polynomial.

$$
\begin{array}{r|l}
 & \quad\ 4x - 1 \\
x^2 - 1 & \ 4x^3 - x^2 - 4x + 1 \\
 & -4x^3 \qquad\ \ + 4x \\ \hline
 & \qquad -x^2 \qquad + 1 \\
 & \qquad\ \ x^2 \qquad - 1 \\ \hline
 & \qquad\qquad\qquad\ \ 0
\end{array}
$$

Reading this:

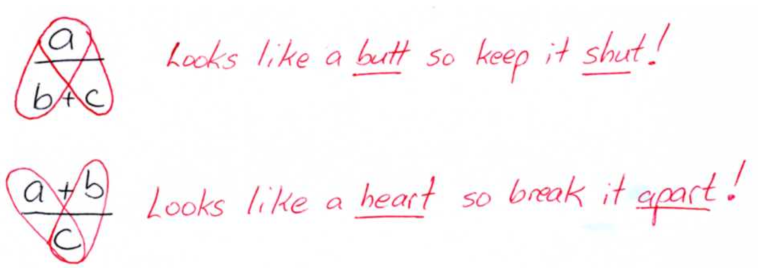
1. Leading terms: since $x^2 \times 4x = 4x^3$, x^2 goes into $4x^3$ a total of $4x$ times.
2. Multiply $4x \times (x^2 - 1) = 4x^3 - 4x$, **which is subtracted by adding $-4x^3 + 4x$** which leaves $-x^2 + 1$.
3. Repeat. Leading terms, since $x^2 \times (-1) = -x^2$, x^2 goes into $-x^2$ a total of -1 times.
4. Multiply $-1 \times (x^2 - 1) = -x^2 + 1$, **which is subtracted by adding $x^2 - 1$** which leaves 0.
5. The remainder is 0, so the division was even.

This computation shows that $\dfrac{4x^3 - x^2 - 4x + 1}{x^2 - 1} = 4x - 1 + \dfrac{0}{x^2 - 1} = 4x - 1, \quad x^2 - 1 \neq 0.$

Aside: Some of you may have learned the process called **synthetic division** to divide two polynomials. You are welcome to use this if you like. I have never used synthetic division in my life, and feel that it obscures what is actually happening when you divide two polynomials. Synthetic division also does not allow you to divide by quantities like $3x^2 + 1$, which long division does without difficulty. Finally, long division of polynomials is much easier for a reader to follow, and therefore much better for us to do! **Some WeBWorK problems may ask you to use synthetic division, just use polynomial long division instead.**

Fun Fact

Someone once showed me the following:



which should help us remember that $\dfrac{a}{b+c} \neq \dfrac{a}{b} + \dfrac{a}{c}$, but we can write $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$.

EXAMPLE Divide $\dfrac{32x^4 + 6x^2 - 1}{8x^2}$.

$$
\begin{aligned}
\frac{32x^4 + 6x^2 - 1}{8x^2} &= \frac{32x^4}{8x^2} + \frac{6x^2}{8x^2} - \frac{1}{8x^2} && \text{(using result above)} \\
&= 4x^2 + \frac{3}{4} - \frac{1}{8}x^{-2} && \text{(cancel common factors, rules of exponents)}
\end{aligned}
$$

3 SI Units

A common use of powers is in SI base unit multipliers, which are shown here for units of mass (g for grams) but can be applied to other measurements such as length (m for meter), time (s for second), temperature (K for kelvin), amount of substance (mol for mole). The more commonly used ones are milli and kilo, which I have highlighted in blue and red.

$10^{-12} = 0.000\,000\,000\,001$	pico (p)	picogram (pg)
$10^{-9} = 0.000\,000\,001$	nano (n)	nanogram (ng)
$10^{-6} = 0.000\,001$	micro (μ)	microgram (μg)
$\mathbf{10^{-3} = 0.001 = \frac{1}{10^3} = \frac{1}{1000}}$	milli (m)	milligram (mg)
$10^{-2} = 0.01 = \frac{1}{10^2} = \frac{1}{100}$	centi (c)	centigram (cg)
$10^{-1} = 0.1 = \frac{1}{10}$	deci (d)	decigram (dg)
$10^{0} = 1$		grams (g)
$10^{1} = 10$	deca (da)	decagram (dag)
$10^{2} = 100 = 10^1 \times 10^1$	hecto (h)	hectogram (hg)
$\mathbf{10^{3} = 1\,000 = 10^1 \times 10^1 \times 10^1}$	kilo (k)	kilogram (kg)
$10^{6} = 1\,000\,000$	mega (M)	megagram (Mg)
$10^{9} = 1\,000\,000\,000$	giga (G)	gigagram (Gg)
$10^{12} = 1\,000\,000\,000\,000$	tera (T)	teragram (Tg)