1 Factoring

Factoring is important since

- it is used to solve some quadratic equations of the form $x^2 + bx + c = 0$ (we will learn another technique involving the quadratic formula later),
- it is used to simplify rational expressions,
- as the complement to the distributive property, it is one of the most important algebraic techniques to effectively work with algebraic expressions,
- it can be used to factor higher degree polynomials and work with more complicated expressions like $8x^3y + 16x^2y^2 - 24x^3y^3$.

1.1 Common factors in terms

Why do this? At the moment, you want the greatest common factor so you can work with smaller numbers:

$$8x^3y + 16x^2y^2 - 24x^3y^3 = \boldsymbol{8x^2y} \cdot x + \boldsymbol{8x^2y} \cdot 2y - \boldsymbol{8x^2y} \cdot 3xy^2 = \boldsymbol{8x^2y}(x + 2y - 3xy^2).$$

EXAMPLE Factor the expression $3x^3 + 9x^2 + 18x$.

SOLUTION:

$$\begin{aligned} 3x^3 + 9x^2 + 18x &= \boldsymbol{3x} \cdot x^2 + \boldsymbol{3x} \cdot 3x + \boldsymbol{3x} \cdot 6 && \text{(identify the common factor in each term)} \\ &= \boldsymbol{3x} \cdot (x^2 + 3x + 6) && \text{(common factor)} \end{aligned}$$

to continue we will need to look at factoring of quadratics.

EXAMPLE Factor the expression $4a^3b - 2a^2b$.

SOLUTION:

$$\begin{aligned} 4a^3b - 2a^2b &= \boldsymbol{2a^2b} \cdot 2a - \boldsymbol{2a^2b} \cdot 1 && \text{(identify the common factor in each term)} \\ &= \boldsymbol{2a^2b} \cdot (2a - 1) && \text{(common factor)} \end{aligned}$$

1.2 Factoring by Grouping

This is just using the distribution property in the other direction:

$$\begin{aligned} (3y - 8)\boldsymbol{(2x - 7)} &= 3y\boldsymbol{(2x - 7)} - 8\boldsymbol{(2x - 7)} && \text{(distribution property)} \\ 3y\boldsymbol{(2x - 7)} - 8\boldsymbol{(2x - 7)} &= (3y - 8)\boldsymbol{(2x - 7)} && \text{(factoring by grouping)} \end{aligned}$$

Notice that factoring by grouping could be thought of as common factoring, since the process is the same. The difference is that the common factor is more than one term, so it is a slightly more complicated looking process, but fundamentally it is just common factoring.

EXAMPLE Factor the expression $10x^3(x - 1) - 15x^2(x - 1)^2$.

SOLUTION:

$$\begin{aligned} 10x^3(x - 1) - 15x^2(x - 1)^2 &= \boldsymbol{5x^2(x - 1)} \cdot 2x - \boldsymbol{5x^2(x - 1)} \cdot 3(x - 1) && \text{(identify common factor)} \\ &= \boldsymbol{5x^2(x - 1)}\big(2x - 3(x - 1)\big) && \text{(common factor)} \\ &= \boldsymbol{5x^2(x - 1)}\big(2x - 3x + 3)\big) && \text{(distribute the } -3\text{)} \\ &= \boldsymbol{5x^2(x - 1)}(-x + 3) && \text{(collect like terms)} \end{aligned}$$

1.3 Factoring trinomials of form $x^2 + bx + c$

$x^2 + bx + c = (x + m)(x + n)$ where m and n are two numbers whose product is c and sum is b.

EXAMPLE Factor $x^2 + 5x - 6$.

SOLUTION: Look for two numbers whose product is -6 and sum is 5: the numbers are $\mathbf{+6}$, $\mathbf{-1}$:

$$x^2 + 5x - 6 = (x + \quad)(x + \quad) = (x \mathbf{+ 6})(x \mathbf{- 1}) = (x + 6)(x - 1)$$

Always multiply out to determine if you have the correct answer:

$$(x + 6)(x - 1) = x(x - 1) + 6(x - 1) = x^2 - x + 6x - 6 = x^2 + 5x - 6.$$

1.4 Factoring trinomials of form $ax^2 + bx + c$

This is more involved, and requires the use of the **trial and error method**, or **the grouping method**. I prefer the grouping method, but both will work. I don't care which one you use.

The Grouping Method to factor trinomials of form $ax^2 + bx + c$

1. Determine the grouping number ac.
2. Find two numbers whose product is ac and sum is b.
3. Use these numbers to write bx as the sum of two terms.
4. Factor by grouping.
5. Check your answer by multiplying out.

EXAMPLE Factor $2x^2 - 7x + 6$.

SOLUTION: Factor by grouping. Find two numbers whose sum is -7 and product is $2 \times 6 = 12$: $-3, -4$.

$$\begin{aligned}
2x^2 \boldsymbol{- 7x} + 6 &= 2x^2 \boldsymbol{- 3x - 4x} + 6 && \text{(rewrite middle term using your two numbers)} \\
&= \left(2x^2 - 3x\right) + \left(-4x + 6\right) && \text{(look for common factor in first pair, last pair)} \\
&= 2x \cdot \boldsymbol{x} - 3 \cdot \boldsymbol{x} + \boldsymbol{(-2)} \cdot 2x - \boldsymbol{(-2)} \cdot 3 && \text{(common factor twice–note minus sign!)} \\
&= (2x - 3)\boldsymbol{x} + \boldsymbol{(-2)}(2x - 3) && \text{(common factor twice)} \\
&= \boldsymbol{(2x - 3)}x - 2\boldsymbol{(2x - 3)} && \text{(factor } 2x - 3 \text{ by grouping)} \\
&= (x - 2)\boldsymbol{(2x - 3)}
\end{aligned}$$

Multiply out to check:

$$(x - 2)(2x - 3) = x(2x - 3) - 2(2x - 3) = 2x^2 - 3x - 4x + 6 = 2x^2 - 7x + 6$$

1.5 How to "Find two numbers whose product is c and sum is b"

1. Trial and error–if the numbers are small enough, you can usually guess your two numbers.
2. A slightly more systematic approach is to list the factors of the product until you either find your two numbers or determine there aren't any (the trinomial is then prime).

For example, if we want two numbers whose product is -154 and whose sum is 15 we can do the following:

Product	Factor	Sum
-154	$(77)(-2)$	75
-154	$(14)(-11)$	3
-154	$(7)(-22)$	-15
-154	$(-7)(22)$	15

Stop once you have found the two numbers, in this case -7 and 22.

Obviously, for large numbers this can be a lot of work. This is one reason why it is suggested that you factor common factors at the beginning, so the numbers you are working with are smaller and the "Find two numbers whose…" step is therefore easier.

> **Advice:** Factoring might take multiple steps. Focus on making each step mathematically correct instead of finding the fastest way to the factorization. There are multiple ways to do each problem, each of which can be correct.

For example, consider the following factorization:

$$\begin{aligned} 5x^2 - 3xy - 10x + 6y, &= x(5x-3y) + 2(-5x+3y), && (1)\\ &= x(5x-3y) - 2(5x-3y), && (2)\\ &= (x-2)(5x-3y). && (3) \end{aligned}$$

Many texts would say that step 1 is wrong because you should have factored -2 out of the last two terms. The text has this completely wrong. How are you supposed to know you should factor -2 instead of 2 at that point? You only know this once you get to step 2 and see that if you factor a -1 out of the second term you will be able to factor $5x - 3y$ out of each term. The solution presented above is mathematically correct at each step and gets to the right answer–there is nothing "wrong" about it!

EXAMPLE Find a polynomial that describes how much greater the perimeter of the square is than the circumference of the circle for a circle that is inscribed in a square with sides of length x. Inscribed means the sides of the circle touch all four sides of the square.

SOLUTION: The perimeter of the square is $x + x + x + x = 4x$.

The circumference of the circle is $2\pi(\text{radius}) = 2\pi\left(\frac{x}{2}\right) = \pi x$.

The difference is $4x - \pi x = (4 - \pi)x$.

EXAMPLE Factor $3x^2 - 33x + 54$.

SOLUTION: Factor the common factor 3 first to get $a = 1$. Two numbers whose product is 18 and sum is -11: -2, -9.

$$\begin{aligned} 3x^2 - 33x + 54 &= 3(x^2 - 11x + 18) \\ &= 3(x-2)(x-9) \end{aligned}$$

Alternate solution: If you didn't notice you could factor a 3 out of each term, you can factor by grouping.

Find two numbers whose product is $\mathbf{3 \times 54 = 162}$ and whose sum is $\mathbf{-33}$: -27 and -6.

Now write the $-33x$ term as two terms based on the numbers you found.

$$\begin{aligned} 3x^2 \boldsymbol{- 33x} + 54 &= 3x^2 \boldsymbol{- 27x - 6x} + 54 \\ &= \left(\boldsymbol{3x^2 - 27x}\right) + \left(\boldsymbol{-6x + 54}\right) \\ &\qquad \text{(blue terms have a common factor of } 3x\text{)} \\ &\qquad \text{(red terms have a common factor of } -6\text{)} \\ &= 3x\boldsymbol{(x-9)} - 6\boldsymbol{(x-9)} \\ &\qquad \text{(both terms have a factor of } x-9\text{)} \\ &= (3x-6)\boldsymbol{(x-9)} \end{aligned}$$

Check: $(3x-6)(x-9) = 3x^2 - 6x - 27x + 54 = 3x^2 - 33x + 54$.

1.6 Special Cases of Factoring

A **prime polynomial** is a polynomial that cannot be factored using the techniques we developed here. For now, we are concerned with integers, so something like $x^2 - 3$ would be considered a prime polynomial even though we can write it as $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$.

Difference of Two Squares

$$a^2 - b^2 = (a-b)(a+b).$$

Perfect Square (two cases)

$$\begin{aligned} a^2 + 2ab + b^2 &= (a+b)^2, \\ a^2 - 2ab + b^2 &= (a-b)^2. \end{aligned}$$

Sum and Difference of Cubes

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2 - ab + b^2), \\ a^3 - b^3 &= (a-b)(a^2 + ab + b^2). \end{aligned}$$

These should be used as formulas you have memorized, not something you "work out". You can "work out" the difference of square and perfect square cases using the grouping method of factoring if you have to, but the sum and difference of cubes must be used as memorized formulas.

> Use the acronym **SOAP** (Same-Opposite-Always Positive) to remember the signs in the cubes formulas.
>
> $$a^3 \text{ sign } b^3 = (a \text{ same sign } b)(a^2 \text{ opposite sign } ab \text{ always positive } b^2)$$

In practice you aren't told which factoring technique to use, so being able to recognize the different cases is important.

1.7 Multiplicity

Multiplicity will be something you see much more of in precalculus, when you look at sketching rational functions.

> **Multiplicity of a factor:** If a factored expression has a factor that appears m times, then the factor has multiplicity m. This means m is the power of the factor.

For example, the perfect square $a^2 + 2ab + b^2 = (a + b)^2$ has a factor of $(a + b)$ with multiplicity 2.

EXAMPLE Factor $27x^3 + \dfrac{729}{8}y^3$.

SOLUTION: Notice that this looks like it might be a sum of cubes. Try to write in terms of quantities cubed:

$$27x^3 + \frac{729}{8}y^3 = (3x)^3 + \left(\frac{9}{2}y\right)^3$$

Now, use the appropriate formula you have memorized: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. Identify: $a = 3x$, $b = \frac{9}{2}y$.

$$\begin{aligned}
27x^3 + \frac{729}{8}y^3 &= (3x)^3 + \left(\frac{9}{2}y\right)^3 \\
&= (a)^3 + (b)^3 && \text{(identify is as sum of cubes)} \\
&= (a + b)(a^2 - ab + b^2) && \text{(use memorized sum of cubes formula)} \\
&= \left(3x + \frac{9}{2}y\right)\left((3x)^2 - (3x)(\frac{9}{2}y) + (\frac{9}{2}y)^2\right) && \text{(sub back in } a = 3x \text{ and } b = \tfrac{9}{2}y\text{)} \\
&= \left(3x + \frac{9}{2}y\right)\left(9x^2 - \frac{27}{2}xy + \frac{81}{4}y^2\right) && \text{(simplify)}
\end{aligned}$$

I would accept the above since it is factored, but remember that you can sometimes write a factored form in different ways. The following are all equivalent:

$$\begin{aligned}
27x^3 + \frac{729}{8}y^3 &= \left(3x + \frac{9}{2}y\right)\left(9x^2 - \frac{27}{2}xy + \frac{81}{4}y^2\right) \\
&= 3\left(x + \frac{3}{2}y\right)\left(9x^2 - \frac{27}{2}xy + \frac{81}{4}y^2\right) \\
&= 3\left(x + \frac{3}{2}y\right)9\left(x^2 - \frac{3}{2}xy + \frac{9}{4}y^2\right) \\
&= 27\left(x + \frac{3}{2}y\right)\left(x^2 - \frac{3}{2}xy + \frac{9}{4}y^2\right) \\
&= 27\left(\frac{2x + 3y}{2}\right)\left(\frac{4x^2 - 6xy + 9y^2}{4}\right) \\
&= \frac{27}{8}(2x + 3y)(4x^2 - 6xy + 9y^2)
\end{aligned}$$

EXAMPLE Factor $9x^2 - 42xy + 49y^2$.

SOLUTION: Notice that the first term and last term look like squares:

$9x^2 = (3x)^2 = a^2$ (so $a = 3x$) and $49y^2 = (7y)^2 = b^2$ (so $b = 7y$).

Check if the middle term is $2ab = 2(3x)(7y) = 42xy$. Yes! So this is a perfect square (difference).

$$
\begin{aligned}
9x^2 - 42xy + 49y^2 &= a^2 - 2ab + b^2 && \text{(identify as a perfect square (difference))}\\
&= (a-b)^2 && \text{(use memorized perfect square (difference) formula)}\\
&= (3x - 7y)^2 && \text{(sub back in } a = 3x \text{ and } b = 7y\text{)}
\end{aligned}
$$

EXAMPLE Factor $\dfrac{1}{256} + \dfrac{11}{40}x + \dfrac{121}{25}x^2$.

SOLUTION: Notice that the first term and last term look like squares:

$\dfrac{1}{256} = \left(\dfrac{1}{16}\right)^2 = a^2$ (so $a = \frac{1}{16}$) and $\dfrac{121}{25}x^2 = \left(\dfrac{11}{5}x\right)^2 = b^2$ (so $b = \frac{11}{5}x$).

Check if the middle term is $2ab = 2\left(\dfrac{1}{16}\right)\left(\dfrac{11}{5}x\right) = \dfrac{11}{40}x$. Yes! So this is a perfect square (sum).

$$
\begin{aligned}
\frac{1}{256} + \frac{11}{40}x + \frac{121}{25}x^2 &= a^2 + 2ab + b^2 && \text{(identify as a perfect square (sum))}\\
&= (a+b)^2 && \text{(use memorized perfect square (sum) formula)}\\
&= \left(\frac{1}{16} + \frac{11}{5}x\right)^2 && \text{(sub back in } a = \tfrac{1}{16} \text{ and } b = \tfrac{11}{5}x\text{)}
\end{aligned}
$$

EXAMPLE Factor $16x^4 - 1$.

SOLUTION:

$$
\begin{aligned}
16x^4 - 1 &= \left(4x^2\right)^2 - (1)^2 && \text{(rewrite to see if it is a difference of squares)}\\
&= (a)^2 - (b)^2 && \text{(identify as difference of squares, } a = 4x^2,\ b = 1\text{)}\\
&= (a+b)(a-b) && \text{(write down memorized formula)}\\
&= (4x^2+1)(4x^2-1) && \text{(substitute back values for } a = 4x^2 \text{ and } b = 1\text{)}\\
&= (4x^2+1)((2x)^2-(1)^2) && \text{(1st polynomial is prime; 2nd is a difference of squares, } a = 2x,\ b = 1\text{)}\\
&= (4x^2+1)(a+b)(a-b) && \text{(write down memorized formula)}\\
&= (4x^2+1)(2x+1)(2x-1) && \text{(substitute back values for } a = 2x \text{ and } b = 1\text{)}
\end{aligned}
$$

EXAMPLE Factor $\frac{1}{16}r^2 - \frac{13}{2}rt + 169t^2$.

SOLUTION: Notice that the first term and last term look like squares:

$\frac{1}{16}r^2 = \left(\frac{1}{4}r\right)^2 = a^2$ (so $a = \frac{1}{4}r$) and $169t^2 = (13t)^2 = b^2$ (so $b = 13t$).

Check if the middle term is $2ab = 2\left(\frac{1}{4}r\right)(13t) = \frac{13}{2}rt$. Yes! So this is a perfect square (difference).

$$\begin{aligned}
\frac{1}{16}r^2 - \frac{13}{2}rt + 169t^2 &= a^2 - 2ab + b^2 && \text{(identify as a perfect square (difference))}\\
&= (a-b)^2 && \text{(use memorized perfect square (difference) formula)}\\
&= \left(\frac{1}{4}r + 13t\right)^2 && \text{(sub back in } a = \tfrac{1}{4}r \text{ and } b = 13t\text{)}
\end{aligned}$$

2 Using Factoring to Solve Quadratic Equations

A very important algebraic rule is the **zero factor property**.

Zero factor property: If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

A common error is set to terms equal to zero (which is not valid). You can only set factors to zero.

$(2x+3) + (x-1) = 0$ does not imply $2x + 3 = 0$ or $x - 1 = 0$ (this has terms, not factors)

$(2x+3) \times (x-1) = 0 \quad \Rightarrow \quad 2x+3 = 0$ or $x - 1 = 0$ (zero factor property).

EXAMPLE

$x^2 - 9 = 0$

$(x-3)(x+3) = 0$ (factoring, difference of squares)

$(x-3) = 0$ or $(x+3) = 0$ (zero factor property)

$x = 3$ or $x = -3$ (simplify)

EXAMPLE

$2x^2 + x - 10 = 0$

Look for two numbers whose product is -20 and sum is $+1$: $+5, -4$

$2x^2 + 5x - 4x - 10 = 0$

$\left(2x^2 + 5x\right) + \left(-4x - 10\right) = 0$ (set up for common factoring)

$x(2x+5) - 2(2x+5) = 0$ (common factoring)

$(x-2)(2x+5) = 0$ (common factoring)

$x - 2 = 0$ or $2x + 5 = 0$ (zero factor property)

$x = 2$ or $x = -5/2$

A very common error in solving quadratic equations $ax^2+bx+c=0$ is to try to "isolate" the x (since this is what you did for linear equations). This is entirely the wrong way to proceed, and will not lead you to the solution.

What this error looks like is the following:

$$
\begin{aligned}
3x^2+5x+2&=0 && \text{(equation we want to solve for } x\text{)}\\
3x^2+5x&=-2\\
x(3x+5)&=-2\\
3x+5&=-\frac{2}{x}\\
3x&=-\frac{2}{x}-5\\
x&=-\frac{2}{3x}-\frac{5}{3}\\
&\vdots
\end{aligned}
$$

and so on. These steps are all mathematically correct, but they will not lead to the solution since your goal when solving a quadratic equation is not to isolate the x. Compare the above to a correct solution that uses factoring:

$$
\begin{aligned}
&3x^2\boldsymbol{+5x}+2=0 && \text{(equation we want to solve for } x\text{)}\\
&\quad\text{(factor by grouping method:)}\\
&\quad\text{(need two numbers whose product is 6 and sum is 5: 2 and 3)}\\
&3x^2\boldsymbol{+2x+3x}+2=0 && \text{(rewrite middle term using 2 and 3)}\\
&\Big(3x^2+2x\Big)+\Big(3x+2\Big)=0 && \text{(set up for common factoring)}\\
&x\boldsymbol{(3x+2)}+\boldsymbol{(3x+2)}=0 && \text{(common factor in terms)}\\
&(x+1)\boldsymbol{(3x+2)}=0 && \text{(factor by grouping)}\\
&(x+1)=0 \text{ or } (3x+2)=0 && \text{(zero factor property)}\\
&x=-1 \text{ or } x=-\frac{2}{3} && \text{(simplify)}
\end{aligned}
$$

Application: Falling Objects When an object is thrown straight upwards, it's height S in meters is given by the quadratic equation

$$S=-\frac{g}{2}t^2+vt+h, \quad \text{where}$$

$$
\begin{aligned}
g &= \text{ acceleration due to gravity, which is } 9.8\text{m/s}^2\\
v &= \text{ initial upward velocity of the object in m/s}\\
h &= \text{ height above ground from which the object is thrown in meters}\\
t &= \text{ time in seconds}
\end{aligned}
$$

This mathematical model of a physical system assumes there is no wind resistance (among other things).

We often approximate the model as $S=-5t^2+vt+h$.

EXAMPLE Jon is in a baseball park outfield stands (right by the field), and he throws a ball straight up in the air with all his might. It leaves his hand with a velocity of 13 m/s, and at a height of 6 m above the field. How long will it take the ball to land on the field?

SOLUTION: The ball hits the field when $S = 0$, and we also have $v = 13$ m/s and $h = 6$ m, so we need to solve

$$0 = -5t^2 + \mathbf{13t} + 6$$

(need two numbers whose product is -30 and sum is 13: 15 and -2)

$$\begin{aligned}
&0 = -5t^2 + \mathbf{15t} - \mathbf{2t} + 6 && \\
&0 = \Big(-5t^2 + 15t\Big) + \Big(-2t + 6\Big) && \text{(set up for common factoring)}\\
&0 = 5t\mathbf{(-t+3)} + 2\mathbf{(-t+3)} && \text{(common factors in terms)}\\
&0 = (5t+2)\mathbf{(-t+3)} && \text{(factor by grouping)}\\
&(5t+2) = 0 \text{ or } (-t+3) = 0 && \text{(zero factor property)}\\
&t = -2/5 \text{ or } t = 3 && \text{(simplify)}
\end{aligned}$$

Exclude the negative answer as unphysical, and we see the ball hits the field after 3 seconds.

EXAMPLE You are standing on a cliff overlooking the ocean. You are 180 meters above the ocean. You drop a pebble into the ocean (dropping implies the initial velocity is 0 m/s). How long will it take for the pebble to hit the water?

SOLUTION: The pebble hits the water when $S = 0$, and we also have $v = 0$ m/s and $h = 180$ m, so we need to solve

$$\begin{aligned}
&0 = -5t^2 + 180 && \\
&0 = -5(t^2 - 36) && \text{(common factors in terms)}\\
&\frac{0}{\mathbf{-5}} = \frac{\cancel{-5}(t^2-36)}{\cancel{\mathbf{-5}}} && \text{(divide each side by } -5\text{)}\\
&0 = t^2 - 36 && \text{(difference of squares)}\\
&0 = (t-6)(t+6) && \\
&(t-6) = 0 \text{ or } (t+6) = 0 && \text{(zero factor property)}\\
&t = 6 \text{ or } t = -6 && \text{(simplify)}
\end{aligned}$$

Exclude the $t = -6$ as unphysical, and we see the pebble hits the water after 6 seconds.

EXAMPLE Factor $x^4 - y^4$.

SOLUTION:

$$\begin{aligned}
x^4 - y^4 &= \big(x^2\big)^2 - (y^2)^2 && \text{(rewrite to see if it is a difference of squares)}\\
&= (a)^2 - (b)^2 && \text{(Identify as difference of squares, } a = x^2,\ b = y^2\text{)}\\
&= (a+b)(a-b) && \text{(write down memorized formula)}\\
&= (x^2+y^2)(x^2-y^2) && \text{(substitute back values for } a = x^2 \text{ and } b = y^2\text{)}\\
&= (x^2+y^2)(x^2-y^2) && \text{(first polynomial is prime; second is a difference of squares, } a = x,\ b = y\text{)}\\
&= (x^2+y^2)(a+b)(a-b) && \text{(write down memorized formula)}\\
&= (x^2+y^2)(x+y)(x-y) && \text{(substitute back values for } a = x \text{ and } b = y\text{)}
\end{aligned}$$

EXAMPLE Factor $3x^3a^3 - 11x^4a^2 - 20x^5a$.

SOLUTION:

$$\begin{aligned} 3x^3a^3 - 11x^4a^2 - 20x^5a &= \boldsymbol{x^3a} \cdot 3a^2 - \boldsymbol{x^3a} \cdot 11xa - \boldsymbol{x^3a} \cdot 20x^2) && \text{(common factors in terms)} \\ &= \boldsymbol{x^3a}(3a^2 - 11xa - 20x^2) && \text{(common factor)} \end{aligned}$$

To factor $3a^2 - 11xa - 20x^2$ we can treat this one of two ways:

1. As a quadratic in x: $-20x^2 - 11ax + 3a^2$
2. As a quadratic in a: $3a^2 - 11xa - 20x^2$

Let's do the first, and see what happens. Let a tag along as if it was a number, and use the grouping method.

The grouping number is $(-20)(3a^2) = -60a^2$.

Find two numbers whose product is $-60a^2$ and whose sum is $-11a$: $-15a$ and $4a$.

We will use these numbers to rewrite the middle term.

$$\begin{aligned} -20x^2 \boldsymbol{- 11xa} + 3a^2 &= -20x^2 \boldsymbol{- 15ax + 4ax} + 3a^2 \\ &= \Big(-20x^2 - 15ax\Big) + \Big(4ax + 3a^2\Big) && \text{(set up for common factoring)} \\ &= -5x\boldsymbol{(4x + 3a)} + a\boldsymbol{(4x + 3a)} && \text{(common factors in terms)} \\ &= (-5x + a)\boldsymbol{(4x + 3a)} && \text{(factor by grouping)} \end{aligned}$$

Putting this back, we have determined that

$$3x^3a^3 - 11x^4a^2 - 20x^5a = x^3a(-5x + a)(4x + 3a).$$

EXAMPLE The area of a triangle is 3 ft^2. The height is 10 ft longer than 4 times the base. Determine the dimensions of the triangle.

SOLUTION: Let x be the length of the base. Then the height is equal to $4x + 10$.

$$\begin{aligned} \text{Area} &= \frac{1}{2}(\text{height})(\text{base}) \\ 3 &= \frac{1}{2}(4x + 10)(x) \\ 6 &= (4x + 10)(x) \\ 4x^2 \boldsymbol{+ 10x} - 6 &= 0 && \text{(Two numbers: product } -24\text{, sum is 10: } 12, -2) \\ 4x^2 \boldsymbol{+ 12x - 2x} - 6 &= 0 \\ \Big(4x^2 + 12x\Big) + \Big(-2x - 6\Big) &= 0 && \text{(set up for common factoring)} \\ 4x\boldsymbol{(x + 3)} - 2\boldsymbol{(x + 3)} &= 0 && \text{(common factors in terms)} \\ (4x - 2)\boldsymbol{(x + 3)} &= 0 && \text{(factor by grouping)} \\ 4x - 2 = 0 &\text{ or } x + 3 = 0 && \text{(zero factor property)} \\ x = 1/2 &\text{ or } x = -3 \end{aligned}$$

Exclude the $x = -3$ as unphysical, and the dimensions of the triangle are base $= 1/2$ ft, and height $= 4(1/2) + 10 = 12$ ft.

EXAMPLE By Dividing $a - b$ into $a^3 - b^3$, verify the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

SOLUTION: Before we can divide, we have to include missing terms: $a^3 - b^3 = a^3 + 0a^2b + 0ab^2 - b^3$.

$$\begin{array}{r l l}
 & a^2 + ab + b^2 & \\
a - b \,\big) & a^3 + 0a^2b + 0ab^2 - b^3 & \\
 & \underline{a^3 - a^2b} & \text{subtract} \\
 & \quad a^2b + 0ab^2 - b^3 & \\
 & \quad \underline{a^2b - ab^2} & \text{subtract} \\
 & \qquad ab^2 - b^3 & \\
 & \qquad \underline{ab^2 - b^3} & \text{subtract} \\
 & \qquad\quad 0 & \text{remainder.}
\end{array}$$

This shows that

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

EXAMPLE Factor $8x^2(x + 1)^3 - 18x^4(x + 1)$.

SOLUTION:

$$\begin{aligned}
8x^2(x+1) - 18x^4(x+1)^3 &= \mathbf{2x^2(x+1)} \cdot 4(x+1)^2 - \mathbf{2x^2(x+1)} \cdot 9x^2 && \text{(identify common factors)} \\
&= \mathbf{2x^2(x+1)}\Big(4(x+1)^2 - 9x^2\Big) && \text{(common factor)} \\
&= 2x^2(x+1)\Big(\mathbf{(2(x+1))}^2 - \mathbf{(3x)}^2\Big) && \text{(difference of squares)} \\
&= 2x^2(x+1)\Big(\mathbf{2(x+1)} + \mathbf{3x}\Big)\Big(\mathbf{2(x+1)} - \mathbf{3x}\Big) && (a^2 - b^2 = (a-b)(a+b)) \\
&= 2x^2(x+1)\Big(2x + 2 + 3x\Big)\Big(2x + 2 - 3x\Big) && \text{(distribute)} \\
&= 2x^2(x+1)\Big(5x + 2\Big)\Big(-x + 2\Big) && \text{(collect like terms)}
\end{aligned}$$

	Original Expression		Factored Expression	
three terms:	$\boxed{8x^3y} + \boxed{16x^2y^2} - \boxed{24x^3y^3}$	$=$	$8x^2y(x+2y-3xy^2)$	four factors
three terms:	$\boxed{3x^3} + \boxed{9x^2} + \boxed{18x}$	$=$	$3x(x^2+3x+6)$	three factors
two terms:	$\boxed{4a^3b} - \boxed{2a^2b}$	$=$	$2a^2b(2a-1)$	four factors
three terms:	$\boxed{x^2} + \boxed{5x} - \boxed{6}$	$=$	$(x+6)(x-1)$	two factors
three terms:	$\boxed{2x^2} - \boxed{7x} + \boxed{6}$	$=$	$(x-2)(2x-3)$	two factors
four terms:	$\boxed{5x^2} - \boxed{3xy} - \boxed{10x} + \boxed{6y}$	$=$	$(x-2)(5x-3y)$	two factors
three terms:	$\boxed{3x^2} - \boxed{33x} + \boxed{54}$	$=$	$(3x-6)(x-9)$	two factors
two terms:	$\boxed{a^2} - \boxed{b^2}$	$=$	$(a+b)(a-b)$	two factors
three terms:	$\boxed{a^2} + \boxed{2ab} + \boxed{b^2}$	$=$	$(a+b)^2$	one factor
three terms:	$\boxed{a^2} - \boxed{2ab} + \boxed{b^2}$	$=$	$(a-b)^2$	one factor
two terms:	$\boxed{a^3} + \boxed{b^3}$	$=$	$(a+b)(a^2-ab+b^2)$	two factors
two terms:	$\boxed{a^3} - \boxed{b^3}$	$=$	$(a-b)(a^2+ab+b^2)$	two factors
two terms:	$\boxed{27x^3} + \boxed{\frac{729}{8}y^3}$	$=$	$\left(3x+\frac{9}{2}y\right)\left(9x^2-\frac{27}{2}xy+\frac{81}{4}y^2\right)$	two factors
three terms:	$\boxed{9x^2} - \boxed{42xy} + \boxed{49y^2}$	$=$	$(3x-7y)^2$	one factor
three terms:	$\boxed{\frac{1}{256}} + \boxed{\frac{11}{40}x} + \boxed{\frac{121}{25}x^2}$	$=$	$\left(\frac{1}{16}+\frac{11}{5}x\right)^2$	one factor
two terms:	$\boxed{16x^4} - \boxed{1}$	$=$	$(4x^2+1)(2x+1)(2x-1)$	three factors
three terms:	$\boxed{\frac{1}{16}r^2} - \boxed{\frac{13}{2}rt} + \boxed{169t^2}$	$=$	$\left(\frac{1}{4}r+13t\right)^2$	one factor
three terms:	$\boxed{3x^3a^3} - \boxed{11x^4a^2} - \boxed{20x^5a}$	$=$	$x^3a(-5x+a)(4x+3a)$	four factors
two terms:	$\boxed{x^4} - \boxed{y^4}$	$=$	$(x^2+y^2)(x+y)(x-y)$	three factors
two terms:	$\boxed{8x^2(x+1)^3} - \boxed{18x^4(x+1)}$	$=$	$2x^2(x+1)(5x+2)(-x+2)$	five factors
three terms:	$\boxed{x^2} + \boxed{3x} + \boxed{6}$	$=$	x^2+3x+6	one factor (prime)

Table 1: The original expressions and equivalent factored expressions seen in this Unit. This is a good place to really nail down the difference between terms and factors. Note the expression $8x^2y$ can be thought of as having three factors, 8, and x^2, and y, however sometimes this is thought of as one factor since it has only one term.