1 Rational Expressions

A **rational expression** is a polynomial divided by another polynomial.

$$\frac{x+y}{14x(y^2-z)}, \qquad \frac{x^2(x+1)}{14(x-2)}, \qquad \frac{x^2+2x+1}{x^2-2x-1}.$$

The denominator in an rational expression cannot equal zero, so exclude values that make a denominator zero.

Manipulating rational expressions involves using the same algebraic rules used for numerical fractions.

1.1 Simplification (simplest form)

Simplification: To simplify rational expressions you must

1. Separately factor the numerator and denominator.
2. Cancel using the rule that you can cancel common factors.

   $\dfrac{a \cdot c}{b \cdot c} = \dfrac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \dfrac{a}{b}$ if $b \neq 0$ and $c \neq 0$.

Note that WeBWorK often does not include the restriction $c \neq 0$ in the solution, instead assuming assuming the restriction is evident in the original expression. I prefer to include the restriction explicitly.

Compare to the similar technique for improper fractions:

$$\begin{aligned}
\frac{2310}{440} &= \frac{(2)(3)(5)(7)(11)}{(2)(2)(2)(5)(11)} && \text{(prime factor the integers)} \\
&= \frac{\cancel{(2)}(3)\cancel{(5)}(7)\cancel{(11)}}{\cancel{(2)}(2)(2)\cancel{(5)}\cancel{(11)}} && \text{(cancel between the numerator and denominator)} \\
&= \frac{(3)(7)}{(2)(2)} \\
&= \frac{21}{4} \\
\frac{2ab}{6ab+4ab^2} &= \frac{2ab}{2ab(3+2b)}, && \text{(common factor the algebraic expressions)} \\
&= \frac{\cancel{2ab}}{\cancel{2ab}(3+2b)}, && \text{(cancel factors between the numerator and denominator)} \\
&= \frac{1}{3+2b}, \qquad 2ab \neq 0 && \text{(note any restrictions on denominators that have been removed)}
\end{aligned}$$

$$\begin{aligned}
\frac{x^2-5x+6}{2x^9-18} &= \frac{(x-2)(x-3)}{2(x-3)(x+3)} && \text{(factor the polynomials)} \\
&= \frac{(x-2)\cancel{(x-3)}}{2\cancel{(x-3)}(x+3)} && \text{(cancel between the numerator and denominator)} \\
&= \frac{(x-2)}{2(x+3)}, \qquad x-3 \neq 0 && \text{(note any restrictions on denominators that have been removed)}
\end{aligned}$$

Advice: This is one of the most commonly misapplied rules of algebra, so make sure you understand it! Misusing this rule will make problems harder to solve than your instructor intends!

EXAMPLE What is wrong with the following work?

$$\frac{(3-x)x+2x+6}{(3-x)(x+1)} = \frac{\cancel{(3-x)}x+2x+6}{\cancel{(3-x)}(x+1)} = \frac{x+2x+6}{x+1} = \frac{3x+6}{x+2}$$

The expression $3-x$ is not a factor of the numerator, and therefore cannot be canceled with the $3-x$ in the denominator.

The correct simplification is:

$$\begin{aligned}
\frac{(3-x)x+2x+6}{(3-x)(x+1)} &= \frac{3x-x^2+2x+6}{(3-x)(x+1)} && \text{(multiply out numerator)}\\
&= \frac{-x^2+5x+6}{(3-x)(x+1)} && \text{(collect like terms)}\\
&= \frac{(6-x)\cancel{(x+1)}}{(3-x)\cancel{(x+1)}} && \text{(factor)}\\
&= \frac{6-x}{3-x}, \quad x+1 \neq 0 && \text{(note any restrictions on denominators that have been removed)}
\end{aligned}$$

1.2 Multiplication

Multiplication:

1. Multiply the numerators and denominators, being careful to use parentheses where needed.

Note that finding a common denominator is not required to multiply algebraic expressions.

$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ if $b \neq 0$ and $d \neq 0$.

Compare to the similar technique for improper fractions: $\frac{5}{46} \times \frac{23}{12} = \frac{5 \times 23}{46 \times 12} = \frac{5 \times \cancel{23}}{2 \times \cancel{23} \times 2^2 \times 3} = \frac{5}{2^3 \times 3}$.

EXAMPLE

$$\begin{aligned}
\frac{3xy}{4} \times \frac{7x}{3y} &= \frac{3 \cdot 7x^2y}{3 \cdot 4 \cdot y}, && \text{(multiply numerators and denominators)}\\
&= \frac{\cancel{3} \cdot 7x^2\cancel{y}}{\cancel{3} \cdot 4 \cdot \cancel{y}}, && \text{(cancel common factors)}\\
&= \frac{7x^2}{4}, \quad y \neq 0. && \text{(note any restrictions on denominators that have been removed)}
\end{aligned}$$

EXAMPLE

$$\begin{aligned}
\frac{x^2-3x+2}{x^2-81}\cdot\frac{x-9}{x^2+5x+6} &= \frac{(x^2-3x+2)(x-9)}{(x^2-81)(x^2+5x+6)} && \text{(multiply numerators and denominators)}\\
&= \frac{(x-2)(x-1)\cancel{(x-9)}}{\cancel{(x-9)}(x+9)(x+2)(x+3)} && \text{(factor, then cancel common factors)}\\
&= \frac{(x-2)(x-1)}{(x+9)(x+2)(x+3)},\quad x-9\neq 0 && \text{(note any restrictions)}
\end{aligned}$$

It is usually best to leave the final expression in its most factored form, but there will be times when you are solving problems when you will then need to multiply everything out. For now, factored is better!

1.3 Division

Division:

1. Multiply by the reciprocal of the quantity you are dividing by.

$\frac{a/b}{c/d} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$ if $b, c, d \neq 0$.

Compare to the similar technique for improper fractions: $\frac{1}{4} \div \frac{7}{12} = \frac{1}{4} \times \frac{12}{7} = \frac{1\times 12}{4\times 7} = \frac{\cancel{4}\times 3}{\cancel{4}\times 7} = \frac{3}{7}$.

EXAMPLE

$$\begin{aligned}
\frac{x/y}{y/x} &= \frac{x}{y}\div\frac{y}{x}, && \text{(rewrite to identify what is quantity being divided by)}\\
&= \frac{x}{y}\cdot\frac{x}{y}, && \text{(multiply by reciprocal)}\\
&= \frac{x^2}{y^2}. && \text{(multiply numerators and denominators)}
\end{aligned}$$

EXAMPLE

$$\begin{aligned}
\frac{x^2+3x+2}{x^2-9}\div\frac{x-3}{x^2+5x+6} &= \frac{x^2+3x+2}{x^2-9}\cdot\frac{x^2+5x+6}{x-3} && \text{(multiply by reciprocal)}\\
&= \frac{(x^2+3x+2)(x^2+5x+6)}{(x^2-9)(x-3)} && \text{(factor)}\\
&= \frac{(x+2)(x+1)(x+2)\cancel{(x+3)}}{\cancel{(x+3)}(x-3)(x-3)} && \text{(cancel)}\\
&= \frac{(x+2)(x+1)(x+2)}{(x-3)(x-3)},\quad x+3\neq 0\\
&= \frac{(x+2)^2(x+1)}{(x-3)^2},\quad x+3\neq 0
\end{aligned}$$

EXAMPLE Simplify $\dfrac{x^2+3x-28}{x^2+14x+49} \div \dfrac{12-3x^2}{x^2+5x-14}$.

We must factor all the polynomials in the expression:

$$x^2+3x-28 = (x+7)(x-4)$$
$$x^2+14x+49 = (x+7)(x+7)$$
$$12-3x^2 = 3(4-x^2) = 3(2-x)(2+x)$$
$$x^2+5x-14 = (x-2)(x+7)$$

$$\frac{x^2+3x-28}{x^2+14x+49} \div \frac{12-3x^2}{x^2+5x-14} = \frac{x^2+3x-28}{x^2+14x+49} \times \frac{x^2+5x-14}{12-3x^2} \quad \text{(division rule)}$$
$$= \frac{(x^2+3x-28)(x^2+5x-14)}{(x^2+14x+49)(12-3x^2)} \quad \text{(multiplication rule)}$$
$$= \frac{\cancel{(x+7)}(x-4)(x-2)\cancel{(x+7)}}{\cancel{(x+7)}\cancel{(x+7)}(3)(2-x)(2+x)} \quad \text{(factor)}$$
$$= \frac{(x-4)(x-2)}{3(2-x)(2+x)}, \quad x+7 \neq 0 \quad \text{(factor } 2-x = -(x-2)\text{)}$$
$$= \frac{(x-4)(x-2)}{-3(x-2)(2+x)}, \quad x+7 \neq 0$$
$$= -\frac{(x-4)\cancel{(x-2)}}{3\cancel{(x-2)}(2+x)}$$
$$= -\frac{(x-4)}{3(2+x)}, \quad x+7 \neq 0, x-2 \neq 0$$

1.4 Addition (and subtraction)

Addition: You must find a lowest common denominator for the expressions.

1. Factor each denominator completely.
2. The LCD (Lowest Common Denominator) is a product containing each different factor.
3. If a factor occurs more than once, the LCD will contain that factor repeated the greatest number of times it occurs in any one denominator.

Compare to the similar technique for improper fractions:

$$\frac{3}{16} - \frac{7}{28} = \frac{3}{2^4} - \frac{7}{2^2 \times 7}, \quad \text{(factor denominators, LCD is } 2^4 \times 7\text{)}$$
$$= \frac{3 \times 7}{2^4 \times 7} - \frac{7 \times 2^2}{2^2 \times 7 \times 2^2}, \quad \text{(write each denominator as LCD)}$$
$$= \frac{3 \times 7 - 7 \times 2^2}{2^4 \times 7}, \quad \text{(add the fractions)}$$
$$= \frac{\cancel{7} \times \left(3 - 2^2\right)}{2^4 \times \cancel{7}}, \quad \text{(factor numerator, cancel common factors)}$$
$$= -\frac{1}{16}. \quad \text{(reduced form)}$$

EXAMPLE

$$\begin{aligned}\frac{8x}{3yz+9z}-\frac{x}{y+3} &= \frac{8x}{3\cdot z\cdot(y+3)}-\frac{x}{(y+3)}, &&\text{(factor denominators, LCD is } 3\cdot z\cdot(y+3))\\ &= \frac{8x}{3\cdot z\cdot(y+3)}-\frac{\mathbf{3\cdot z\cdot}x}{\mathbf{3\cdot z\cdot}(y+3)}, &&\text{(write each denominator as LCD)}\\ &= \frac{8x-3\cdot z\cdot x}{3\cdot z\cdot(y+3)}, &&\text{(add, now that denominators are same)}\\ &= \frac{x(8-3z)}{3z(y+3)}. &&\text{(factor numerator)}\end{aligned}$$

EXAMPLE Simplify $\frac{x-8}{x^2-4x+3}+\frac{x+2}{x^2-1}$.

Factor everything:

$$\begin{aligned}x^2-4x+3 &= (x-3)(x-1)\\ x^2-1 &= (x-1)(x+1)\end{aligned}$$

Start simplifying:

$$\begin{aligned}\frac{x-8}{x^2-4x+3}+\frac{x+2}{x^2-1} &= \frac{x-8}{(x-3)(x-1)}+\frac{x+2}{(x-1)(x+1)} &&\text{(LCD is } (x-3)(x-1)(x+1))\\ &= \frac{(x-8)\mathbf{(x+1)}}{(x-3)(x-1)\mathbf{(x+1)}}+\frac{(x+2)\mathbf{(x-3)}}{(x-1)(x+1)\mathbf{(x-3)}} &&\text{(get common denominator)}\\ &= \frac{(x-8)(x+1)+(x+2)(x-3)}{(x-3)(x-1)(x+1)} &&\text{(add)}\\ &= \frac{x^2-7x-8+x^2-x-6}{(x-3)(x-1)(x+1)} &&\text{(multiply out)}\\ &= \frac{2x^2-8x-14}{(x-3)(x-1)(x+1)} &&\text{(collect like terms)}\\ &= \frac{2(x^2-4x-7)}{(x-3)(x-1)(x+1)} &&\text{(numerator is prime polynomial)}\end{aligned}$$

EXAMPLE Simplify $\frac{x-4}{x^2-4x+3}-\frac{x+2}{x^2-1}$.

Factor everything:

$$\begin{aligned}x^2-4x+3 &= (x-3)(x-1)\\ x^2-1 &= (x-1)(x+1)\end{aligned}$$

Start simplifying:

$$\begin{aligned}
\frac{x-4}{x^2-4x+3} - \frac{x+2}{x^2-1} &= \frac{x-4}{(x-3)(x-1)} - \frac{x+2}{(x-1)(x+1)} && \text{(LCD is } (x-3)(x-1)(x+1)\text{)}\\
&= \frac{(x-4)\boldsymbol{(x+1)}}{(x-3)(x-1)\boldsymbol{(x+1)}} - \frac{(x+2)\boldsymbol{(x-3)}}{(x-1)(x+1)\boldsymbol{(x-3)}} && \text{(get common denominator)}\\
&= \frac{(x-4)(x+1)-(x+2)(x-3)}{(x-3)(x-1)(x+1)} && \text{(add)}\\
&= \frac{x^2-3x-4-(x^2-x-6)}{(x-3)(x-1)(x+1)} && \text{(simplify numerator)}\\
&= \frac{x^2-3x-4-x^2+x+6}{(x-3)(x-1)(x+1)} && \text{(distribute minus sign)}\\
&= \frac{-2x+2}{(x-3)(x-1)(x+1)} && \text{(collect like terms)}\\
&= \frac{-2\cancel{(x-1)}}{(x-3)\cancel{(x-1)}(x+1)} && \text{(cancel common factors)}\\
&= \frac{-2}{(x-3)(x+1)}, \quad x-1 \neq 0
\end{aligned}$$

An Application

We can express the average velocity of the object over a time interval by the formula:

$$\text{Average Velocity from } t=t_1 \text{ to } t=t_2 = \frac{\text{distance traveled}}{\text{change in time}}$$

If we drop a ball from rest (meaning initial velocity is zero) at a height h meters above ground, then the height above ground S is approximated by the following equation:

$$S = -5t^2 + h,$$

where $h =$ height above ground from which the object is dropped in meters
$t =$ time in seconds

The average velocity of the ball during time interval from t_1 to t_2 is given by

$$\begin{aligned}
\text{Average Velocity from } t=t_1 \text{ to } t=t_2 &= \frac{\text{distance traveled}}{\text{change in time}}\\
&= \frac{(-5t_2^2+h)-(-5t_1^2+h)}{t_2-t_1}\\
&= \frac{-5t_2^2+h+5t_1^2-h}{t_2-t_1}\\
&= \frac{-5(t_2^2-t_1^2)}{t_2-t_1}\\
&= \frac{-5(t_2+t_1)(t_2-t_1)}{t_2-t_1}\\
&= -5(t_2+t_1) \text{ meters/second}, \quad t_2 \neq t_1
\end{aligned}$$

Notice that factoring and simplifying lead to a much simpler expression for the average velocity in the end. The result is negative since the ball is moving downwards, towards the earth. To treat the case when $t_2 = t_1$ requires calculus.

1.5 Complex Rational Expressions

If you have a **complex rational expression** (ie., you have multiple denominators), get a common denominator in the overall numerator and denominator, then use the rules for division to simplify. An example helps make the process clear. EXAMPLE In this example there is a main numerator with LCD of xy, and a main denominator with LCD of x^2y.

$$\begin{aligned}
\frac{\frac{1-x}{x}+\frac{x}{y}}{\frac{1}{x^2y}-\frac{y}{x}} &= \frac{\frac{1-x}{x}\cdot\frac{y}{y}+\frac{x}{y}\cdot\frac{x}{x}}{\frac{1}{x^2y}-\frac{y}{x}\cdot\frac{xy}{xy}} && \text{(write with LCDs)}\\
&= \frac{\frac{(1-x)y}{xy}+\frac{x^2}{xy}}{\frac{1}{x^2y}-\frac{xy^2}{x^2y}} && \text{(add now that denominators are same)}\\
&= \frac{\left(\frac{y-xy+x^2}{xy}\right)}{\left(\frac{1-xy^2}{x^2y}\right)}\\
&= \left(\frac{y-xy+x^2}{\cancel{xy}}\right)\cdot\left(\frac{x^{\cancel{2}}\cancel{y}}{1-xy^2}\right) && \text{(multiplying by reciprocal, cancel common factors)}\\
&= \left(y-xy+x^2\right)\cdot\left(\frac{x}{1-xy^2}\right), \quad x\neq 0, y\neq 0\\
&= \frac{(y-xy+x^2)x}{1-xy^2}, \quad x\neq 0, y\neq 0
\end{aligned}$$

EXAMPLE Simplify the following expression: $\frac{\frac{5}{x+4}}{\frac{1}{x-4}-\frac{2}{x^2-16}}$.

$$\begin{aligned}
\frac{\frac{5}{x+4}}{\frac{1}{x-4}-\frac{2}{x^2-16}} &= \frac{\left(\frac{5}{x+4}\right)}{\frac{1}{x-4}\cdot\frac{x+4}{x+4}-\frac{2}{(x+4)(x-4)}} && \text{(factor, get LCD } (x+4)(x-4) \text{ for main denominator)}\\
&= \frac{\left(\frac{5}{x+4}\right)}{\frac{x+4}{(x+4)(x-4)}-\frac{2}{(x+4)(x-4)}} && \text{(add now that we have LCD)}\\
&= \frac{\left(\frac{5}{x+4}\right)}{\frac{x+4-2}{(x+4)(x-4)}}\\
&= \left(\frac{5}{x+4}\right)\times\frac{(x+4)(x-4)}{x+2} && \text{(divide: multiply by reciprocal)}\\
&= \frac{5\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(x+2)} && \text{(cancel common factors)}\\
&= \frac{5(x-4)}{x+2}, \quad x+4\neq 0
\end{aligned}$$

2 Solving Rational Equations

When solving equations involving rational expressions, the following technique always works.

To Solve a Rational Equation:

1. Determine the LCD of all the denominators.
2. Multiply each term in the equation by the the LCD.
3. Solve the resulting equation.
4. Check the solution–you should exclude any solution that you find which makes the LCD zero (it would result in division by zero in the original equation, so it is not allowed). These excluded solutions are called **extraneous solutions**.

EXAMPLE Solve $\dfrac{x+11}{x^2-5x+4}+\dfrac{3}{x-1}=\dfrac{5}{x-4}$.

Factor x^2-5x+4: Need two numbers whose product is 4 and sum is -5: $-4, -1$.

$$x^2-5x+4=(x-4)(x-1).$$

The LCD for the the equation is $\mathbf{(x-4)(x-1)}$. Multiply all terms in the equation by this LCD:

$$\frac{x+11}{x^2-5x+4}+\frac{3}{x-1}=\frac{5}{x-4}$$
$$\frac{x+11}{\cancel{(x-4)}\cancel{(x-1)}}\cdot\cancel{(x-4)}\cancel{(x-1)}+\frac{3}{\cancel{x-1}}\cdot(x-4)\cancel{(x-1)}=\frac{5}{\cancel{x-4}}\cdot\cancel{(x-4)}(x-1)$$
$$x+11+3(x-4)=5(x-1)$$
$$x+11+3x-12=5x-5$$
$$4x-1=5x-5$$
$$4x-5x=-5+1$$
$$-x=-4$$
$$x=4$$

We aren't done until we verify this is actually a solution. Since $x=4$ makes the LCD zero, this is not a solution since it would result in division by zero.

Therefore, $x=4$ is an extraneous solution (meaning it is not a solution), and the original equation has no solution.

Solving equations is obviously important, so make sure you understand the techniques for rational equations. You should be very adept at solving equations like these.

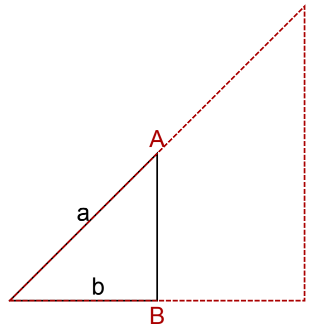
3 Proportion

To Solve a Proportion Problem:

1. Organize the information you are given.
2. Write a proportion equating the respective parts, with x as the unknown quantity. Units should cancel.
3. Solve for x.

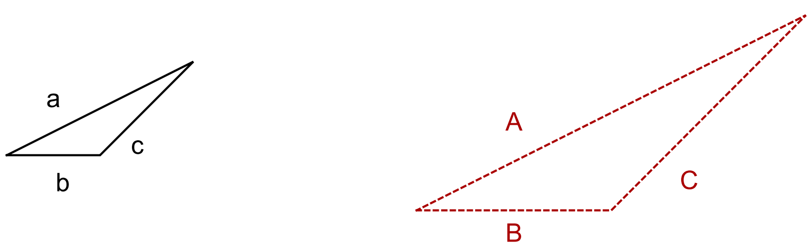
3.1 Similar Triangles

Similar Triangles have ratios of lengths that are equal. For example:



For the above pair of similar triangles, it is true that $\dfrac{a}{A} = \dfrac{b}{B}$.

Here are some more general similar triangles:



In general, we can write a variety of ratios (we could have done this for the right triangles above as well):

$$\frac{a}{A} = \frac{c}{C}, \qquad \frac{a}{A} = \frac{b}{B}, \qquad \frac{b}{B} = \frac{c}{C}.$$

We get the same ratios if we put the quantities from one triangle on the left, and the other triangle on the right:

$$\frac{a}{c} = \frac{A}{C}, \qquad \frac{a}{b} = \frac{A}{B}, \qquad \frac{b}{c} = \frac{B}{C}.$$

Proportion is something that is incredibly useful in day-to-day life. The trick to getting the ratios set up correctly is to include the units, and make sure the units cancel off. For similar triangles (where the units are the same), you can put the quantities from one triangle in the numerator, and quantities from the other triangle in the denominator. There are other ways to do this correctly, this is just my suggestion for getting the ratio correct.

EXAMPLE Tim is driving his U-Haul truck, and has to hit the brakes while traveling at 55 mph due to heavy traffic. If he slows at a rate of 2 mph for every 3 seconds he is braking, how fast will be be traveling 10 seconds after he applied the brakes (assuming he brakes the entire time)?

This is a proportion problem.

$$\frac{\text{time spent braking}}{\text{reduction in speed}} \longrightarrow \quad \frac{3 \text{ sec}}{2 \text{ mph}} = \frac{10 \text{ sec}}{x \text{ mph}} \quad \longleftarrow \frac{\text{given time spent braking}}{\text{unknown reduction in speed}}$$

$$\frac{3}{2} = \frac{10}{x} \quad \text{(notice the units cancel)}$$

$$x = 10 \cdot \frac{2}{3} = 6.67 \text{ mph} \quad \text{(solve for } x\text{)}$$

Now, this is the reduction in Tim's speed, so he is traveling $55 - 6.67 = 48.33$ mph after 10 seconds.

EXAMPLE It takes Rhonda 150 minutes to clean the house. It takes Pauline 100 minutes to clean the house. Working together, how long will it take them to clean the house?

In 1 minute Rhonda will finish $\frac{1}{150}$ of the cleaning.

In 1 minute Pauline will finish $\frac{1}{100}$ of the cleaning

Let x be the time it takes them to complete the cleaning together.

In 1 minute together they will finish $\frac{1}{x}$ of the cleaning, which means in 1 minute:

(amount Rhonda Cleans) + (amount Pauline Cleans) = (amount they clean together)

$$\frac{1}{150} + \frac{1}{100} = \frac{1}{x}$$

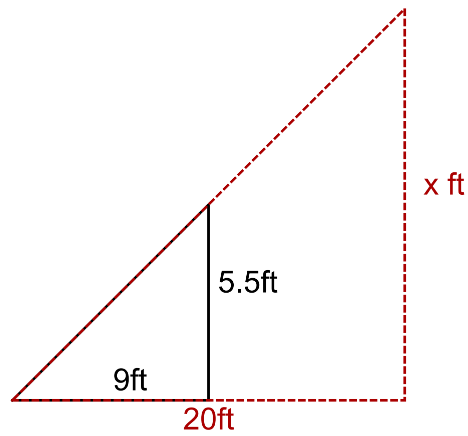
$$\frac{2}{300} + \frac{3}{300} = \frac{1}{x}$$

$$\frac{5}{300} = \frac{1}{x}$$

$$x = 60$$

It will take 60 minutes to clean the house if they work together.

EXAMPLE Sam is 5.5 ft tall and casts a shadow of 9 ft. At the same moment, the statue she is admiring in the park casts a shadow of 20 ft. How tall is the statue?



Let x be the height of the statue. Then in the diagram above the dashed red triangle represents the statue and the shadow it casts, the black triangle represents Sam and the shadow she casts.

From the diagram:

$$\frac{9\text{ft}}{20\text{ft}} = \frac{5.5\text{ft}}{x\text{ft}}$$
$$x = \frac{20}{9}(5.5) = 12.22$$

So the statue is 12.22ft high.

EXAMPLE In a sample survey of 145 people in Morris, it was found that 15 people of the 145 surveyed missed the Burger King that used to be here. If the population of Morris is 5,000 people, what does this survey predict is the number of people in Morris who miss the Burger King?

$$\frac{\text{Number of People in Survey}}{\text{Number of People in Survey who Miss BK}} = \frac{\text{Number of People in Morris}}{\text{Number of People in Morris who Miss BK}}$$
$$\frac{145}{15} = \frac{5000}{\text{Number of People in Morris who Miss BK}}$$
$$\text{Number of People in Morris who Miss BK} = \frac{15}{145}(5000)$$
$$= \frac{15}{145}(5000) = 517.2$$

The survey (assuming it was a good survey, which is another question altogether) predicts that 517 people in Morris miss Burger King.