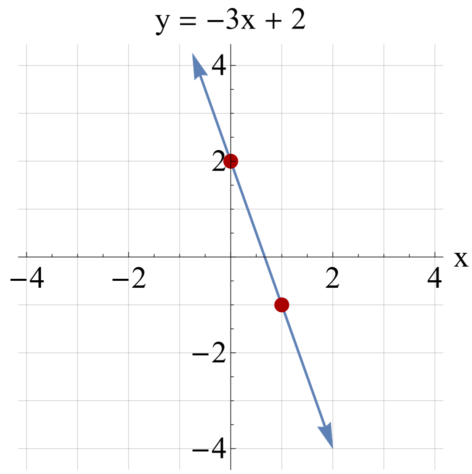
1 Relations

We saw earlier linear equations plots in the Cartesian plane.

EXAMPLE Sketch $y = -3x + 2$.

If $x = 0$, the the equation becomes $y = -3(0) + 2 \Rightarrow y = 2$, so an ordered pair on the line is $(x, y) = (0, 2)$.

If $x = 1$, the the equation becomes $y = -3(1) + 2 \Rightarrow y = -1$, so an ordered pair on the line is $(x, y) = (1, -1)$.



The ordered pairs $(x, y) = (x, -3x + 2)$ represent a relationship between an **independent variable**, x, and a **dependent variable**, y (since the value of y depends on the value of x you choose to evaluate at).

We call any set of ordered pairs a **relation**.

The first coordinates of the ordered pairs in a relation are called the **domain**.

The second coordinates of the ordered pairs in a relation are called the **range**.

For the example relation $(x, -3x + 2)$, the

- domain is $x \in (-\infty, \infty)$ (using interval notation) or $-\infty < x < \infty$ (in set notation).
- range is $y \in (-\infty, \infty)$ (using interval notation) or $-\infty < y < \infty$ (in set notation).

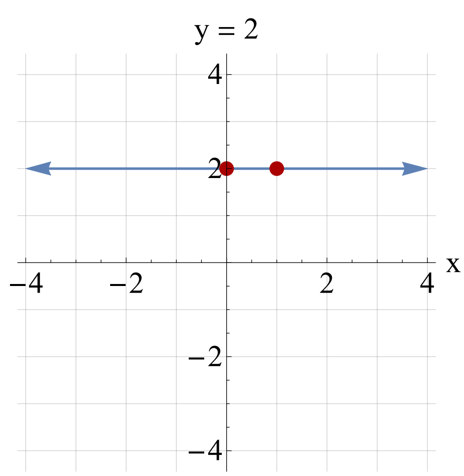
Here are some examples of relations, you can see that they can be quite varied!

EXAMPLE Relation: $(x, 2) = (x, 2)$

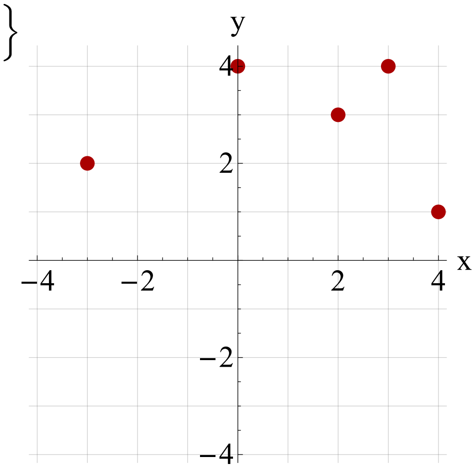
Note this relation is the same as saying $y = 2$, a horizontal line.

Domain: $-\infty < x < \infty$

Range: $y = 2$ or $\{2\}$ using set notation.



EXAMPLE Relation: $\Big\{(\mathbf{-3}, \mathbf{2}), (\mathbf{-1}, \mathbf{2}), (\mathbf{0}, \mathbf{4}), (\mathbf{2}, \mathbf{3}), (\mathbf{3}, \mathbf{4}), (\mathbf{4}, \mathbf{1})\Big\}$

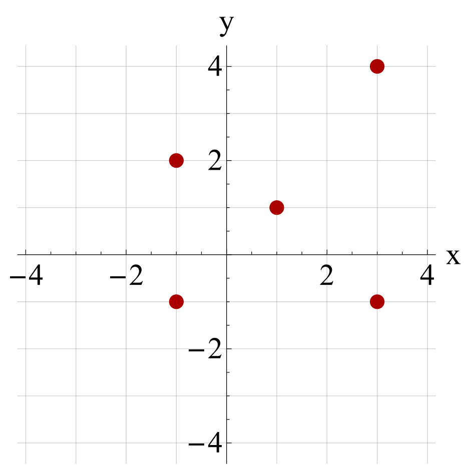


Note this relation is just a list of ordered pairs.

Domain: $\Big\{\mathbf{-3, -1, 0, 2, 3, 4}\Big\}$

Range: $\Big\{\mathbf{1, 2, 3, 4}\Big\}$

EXAMPLE Relation: $\Big\{(\mathbf{-1}, \mathbf{2}), (\mathbf{-1}, \mathbf{-1}), (\mathbf{1}, \mathbf{1}), (\mathbf{3}, \mathbf{4}), (\mathbf{3}, \mathbf{-1})\Big\}$

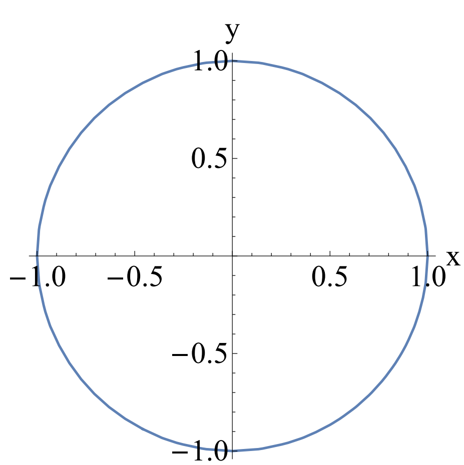


Note this relation is just a list of ordered pairs.

Domain: $\Big\{\mathbf{-1, 1, 3}\Big\}$

Range: $\Big\{\mathbf{-1, 1, 2, 4}\Big\}$

EXAMPLE Relation: $x^2 + y^2 = 1$.



Note this relation is an equation that the ordered pairs must satisfy.

Domain: $-1 \le x \le 1$ (from looking at the graph)

Range: $-1 \le y \le 1$ (from looking at the graph)

We see in the above examples that relations could be

- a discrete set of ordered pairs, like $\Big\{(-1,2),(-1,-1),(1,1),(3,4),(3,-1)\Big\}$ or
- an equation that the ordered pairs must satisfy, like $x^2+y^2=1$ or
- a relation of the form $(x, f(x))$, like $(x, -3x+2)$.

The latter form is something special, and we will focus on that form next.

2 Functions and Their Properties

A **function** is a relation where no two different ordered pair have the same first coordinate.

In our previous examples of relations, we have:

$y = -2x + 2$ (a function)

$y = 2$ (a function)

$\Big\{(-3,2),(-1,2),(0,4),(2,3),(3,4),(4,1)\Big\}$ (a function)

Two of the relations are not functions.

The set that is not a function has two ordered pairs with the same x value and different y values.

$\Big\{\mathbf{(-1,2)},\mathbf{(-1,-1)},(1,1),\mathbf{(3,4)},\mathbf{(3,-1)}\Big\}$ (not a function)

The equation $x^2+y^2=1$ is satisfied by two ordered pairs $(0,1)$ and $(0,-1)$, so the same x value and different y values.

$x^2+y^2=1$ (not a function)

2.1 The Linear Function and Functional Notation

Since the function $(x, f(x))$ represents a set of ordered pairs (x, y), we can get the function notation $y = f(x)$ by comparing the two different ways of writing the ordered pair for a function.

This means we can represent the linear equation $y = mx + b$ as a function $f(x) = mx + b$.

To use the functional notation properly, you want to think of the different pieces of the notation and what the mean.

Functional Notation: $y = f(x)$

1. The functional notation uses x as a placeholder for an element from the domain, and $f(x)$ (read as "f of x") refers to the associated value in the range.
2. The function $y = f(x)$ represents a set of points in the Cartesian plane.
3. The notation $f(x)$ **does not mean multiplication,** i.e., $f(x) \neq f \times x$.

Examples of Proper Functional Evaluation

What we need to be able to do is evaluate the function at different elements from the domain correctly.

Functional Evaluation:

Step 1. Rewrite the function so it is clear what you started with.

Step 2. Rewrite the function, this time replacing every x with brackets (do not simplify yet).

Step 3. The function is now ready to accept any input from the domain. Put the quantity you want the function to operate on in all the brackets (do not simplify yet).

Step 4. Now that the function has been evaluated and given us the correct output, we may proceed with any algebraic simplification.

I am going to show all the steps here, this is a place many people make mistakes but if you are careful you can save yourself much frustration! **Advice: Follow this process until you are comfortable with functional notation.**

EXAMPLE Given $f(x) = \dfrac{x-1}{1-x^2}$, evaluate $f(\frac{1}{2})$.

Step 1. Rewrite the function so it is clear what you started with.

$$f(x) = \frac{x-1}{1-x^2}.$$

Step 2. Rewrite the function, this time replacing every x with brackets (do not simplify yet).

$$f(\quad) = \frac{(\quad)-1}{1-(\quad)^2}.$$

Step 3. The function is now ready to accept any input from the domain. Put the quantity you want the function to operate on in all the brackets (do not simplify yet).

$$f\left(\mathbf{1/2}\right) = \frac{(\mathbf{\frac{1}{2}})-1}{1-(\mathbf{\frac{1}{2}})^2}.$$

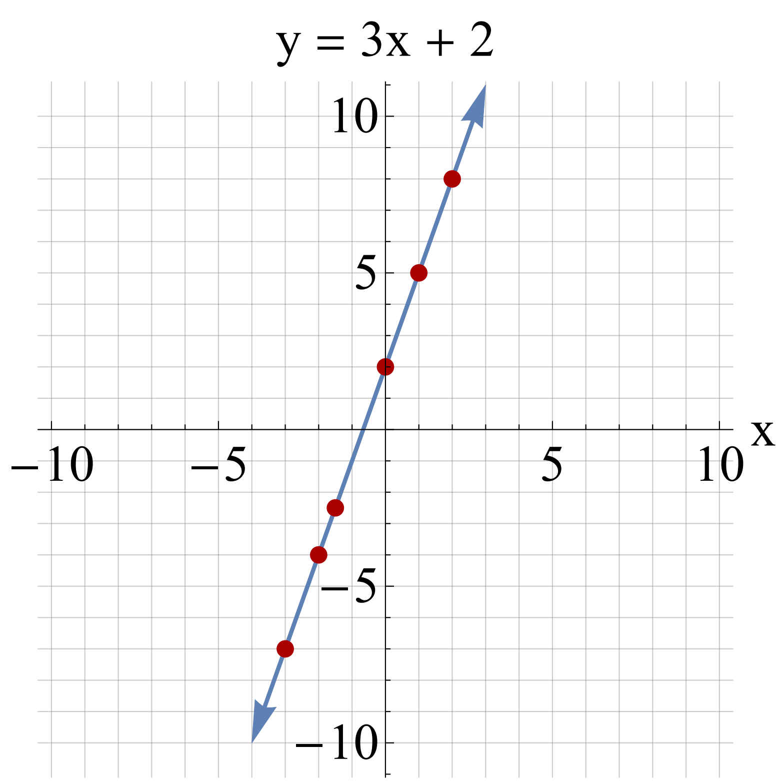
Step 4. Now that the function has been evaluated and given us the correct output, we may proceed with any algebraic simplification.

$$\begin{aligned} f(\mathbf{1/2}) &= \frac{(\mathbf{\frac{1}{2}})-1}{1-(\mathbf{\frac{1}{2}})^2}, && \text{(functional substitution)} \\ &= \frac{\left(-\frac{1}{2}\right)}{\left(\frac{3}{4}\right)} && \text{(simplify fractions)} \\ &= \left(-\frac{1}{2}\right)\cdot\left(\frac{4}{3}\right) && \text{(invert and multiply)} \\ &= -\frac{2}{3} \end{aligned}$$

EXAMPLE Given the function $f(x) = 3x + 2$, then what is the value of the function f acting on the elements from the domain: -3, -2, $-3/2$, 0, 1, 2.

$$f(\mathbf{-3}) = 3(\mathbf{-3}) + 2 = -7 \quad \Rightarrow \quad \text{ordered pair is } (-3, -7)$$
$$f(\mathbf{-2}) = 3(\mathbf{-2}) + 2 = -4 \quad \Rightarrow \quad \text{ordered pair is } (-2, -4)$$
$$f(\mathbf{-3/2}) = 3(\mathbf{-3/2}) + 2 = -9/2 + 4/2 = -5/2 \quad \Rightarrow \quad \text{ordered pair is } (-3/2, -5/2)$$
$$f(\mathbf{0}) = 3(\mathbf{0}) + 2 = 2 \quad \Rightarrow \quad \text{ordered pair is } (0, 2)$$
$$f(\mathbf{1}) = 3(\mathbf{1}) + 2 = 5 \quad \Rightarrow \quad \text{ordered pair is } (1, 5)$$
$$f(\mathbf{2}) = 3(\mathbf{2}) + 2 = 8 \quad \Rightarrow \quad \text{ordered pair is } (2, 8)$$

Here is a plot of the points we have just found for the function. The line represents the other ordered pairs that we did not compute.



Domain: $-\infty < x < \infty$ or in interval notation $(-\infty, \infty)$.

Range: $-\infty < y < \infty$ or in interval notation $(-\infty, \infty)$.

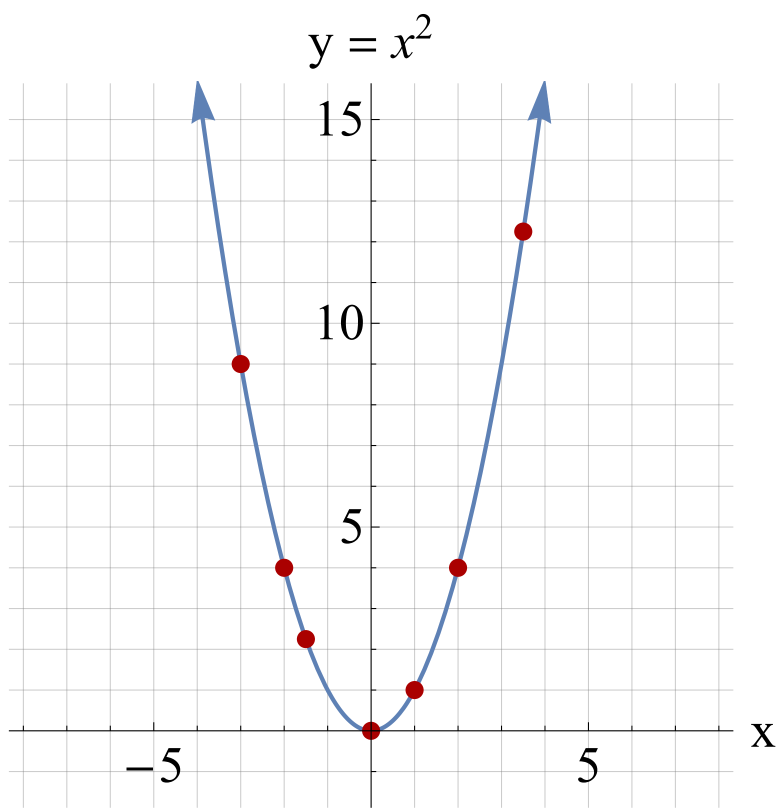
This process of finding ordered pairs can be used to sketch more complicated nonlinear functions.

EXAMPLE Given the function $f(x) = x^2$, then what is the value of the function f acting on the elements from the domain: -3, -2, $-3/2$, 0, 1, 2, $7/2$.

Note $f(x) = x^2$ is a **quadratic function**, which we will examine in more detail in Unit 12.

$$
\begin{aligned}
f(\mathbf{-3}) &= (\mathbf{-3})^2 = 9 && \Rightarrow && \text{ordered pair is } (-3, 9)\\
f(\mathbf{-2}) &= (\mathbf{-2})^2 = 4 && \Rightarrow && \text{ordered pair is } (-2, 4)\\
f(\mathbf{-3/2}) &= (\mathbf{-3/2})^2 = 9/4 && \Rightarrow && \text{ordered pair is } (-3/2, 9/4)\\
f(\mathbf{0}) &= (\mathbf{0})^2 = 0 && \Rightarrow && \text{ordered pair is } (0, 0)\\
f(\mathbf{1}) &= (\mathbf{1})^2 = 1 && \Rightarrow && \text{ordered pair is } (1, 1)\\
f(\mathbf{2}) &= (\mathbf{2})^2 = 4 && \Rightarrow && \text{ordered pair is } (2, 4)\\
f(\mathbf{7/2}) &= (\mathbf{7/2})^2 = 49/4 && \Rightarrow && \text{ordered pair is } (7/2, 49/4)
\end{aligned}
$$

Here is a plot of the points we have just found for the function. The line represents the other ordered pairs that we did not compute.



Domain: $-\infty < x < \infty$ or in interval notation $(-\infty, \infty)$.

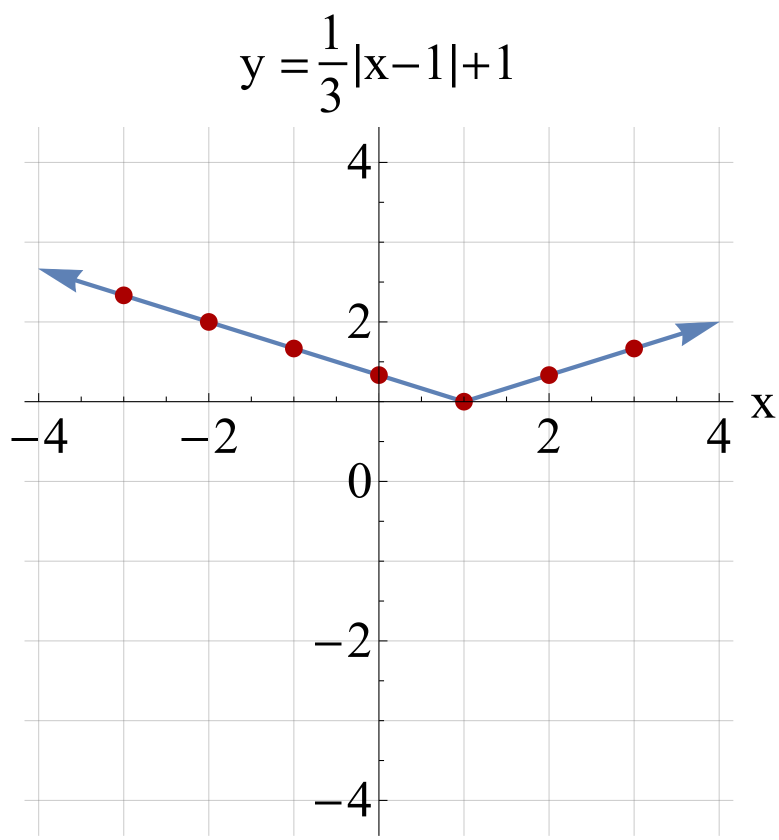
Range: $0 \leq y < \infty$ or in interval notation $[0, \infty)$.

EXAMPLE Given the function $f(x) = \frac{1}{3}\big|x-1\big|+1$, then what is the value of the function f acting on the elements from the domain: -3, -2, 1, 0, 1, 2, 3.

Note $f(x) = \big|x\big|$ is the **absolute value function**, so this is a modification of that function.

$$\begin{aligned}
f(\mathbf{-3}) &= \frac{1}{3}\big|(\mathbf{-3})-1\big|+1 = \frac{1}{3}\big|-4\big|+1 = \frac{1}{3}(4)+1 = \frac{7}{3} && \Rightarrow && \text{ordered pair is } (-3, 7/3)\\
f(\mathbf{-2}) &= \frac{1}{3}\big|(\mathbf{-2})-1\big|+1 = \frac{1}{3}\big|-3\big|+1 = \frac{1}{3}(3)+1 = 2 && \Rightarrow && \text{ordered pair is } (-2, 2)\\
f(\mathbf{-1}) &= \frac{1}{3}\big|(\mathbf{-1})-1\big|+1 = \frac{1}{3}\big|-2\big|+1 = \frac{1}{3}(2)+1 = \frac{5}{3} && \Rightarrow && \text{ordered pair is } (-1, 5/3)\\
f(\mathbf{0}) &= \frac{1}{3}\big|(\mathbf{0})-1\big|+1 = \frac{1}{3}\big|-1\big|+1 = \frac{1}{3}(1)+1 = \frac{4}{3} && \Rightarrow && \text{ordered pair is } (0, 4/3)\\
f(\mathbf{1}) &= \frac{1}{3}\big|(\mathbf{1})-1\big|+1 = \frac{1}{3}\big|0\big|+1 = \frac{1}{3}(0)+1 = 1 && \Rightarrow && \text{ordered pair is } (1, 1)\\
f(\mathbf{2}) &= \frac{1}{3}\big|(\mathbf{2})-1\big|+1 = \frac{1}{3}\big|1\big|+1 = \frac{1}{3}(1)+1 = \frac{4}{3} && \Rightarrow && \text{ordered pair is } (2, 4/3)\\
f(\mathbf{3}) &= \frac{1}{3}\big|(\mathbf{3})-1\big|+1 = \frac{1}{3}\big|2\big|+1 = \frac{1}{3}(2)+1 = \frac{5}{3} && \Rightarrow && \text{ordered pair is } (3, 5/3)
\end{aligned}$$

Here is a plot of the points we have just found for the function. The line represents the other ordered pairs that we did not compute.



Domain: $-\infty < x < \infty$ or in interval notation $(-\infty, \infty)$.

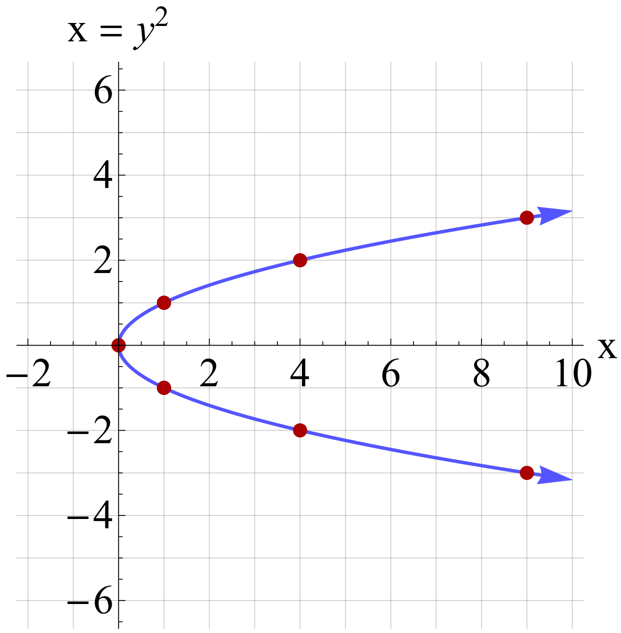
Range: $0 \leq y < \infty$ or in interval notation $[0, \infty)$.

This process of determining some ordered pairs even works in some instances when we have a relation that is not a function. Note the example below the y values are used to create the ordered pairs, and the resulting graph has two values for a given x value so the relation $x = y^2$ is not a function.

EXAMPLE Given the relation $x = y^2$, then what is the value of the ordered pairs when $y = -3$, $y = -2$, $y = -1$, $y = 0$, $y = 1$, $y = 2$, $y = 3$.

$y = -3$ then $x = (\mathbf{-3})^2 = 9$ $\Rightarrow$ ordered pair is $(9, -3)$
$y = -2$ then $x = (\mathbf{-2})^2 = 4$ $\Rightarrow$ ordered pair is $(4, -2)$
$y = -1$ then $x = (\mathbf{-1})^2 = 1$ $\Rightarrow$ ordered pair is $(1, -1)$
$y = 0$ then $x = (\mathbf{0})^2 = 0$ $\Rightarrow$ ordered pair is $(0, 0)$
$y = 1$ then $x = (\mathbf{1})^2 = 1$ $\Rightarrow$ ordered pair is $(1, 1)$
$y = 2$ then $x = (\mathbf{2})^2 = 4$ $\Rightarrow$ ordered pair is $(4, 2)$
$y = 3$ then $x = (\mathbf{3})^2 = 9$ $\Rightarrow$ ordered pair is $(9, 3)$

Here is a plot of the points we have just found for the function. The line represents the other ordered pairs that we did not compute.



Domain: $0 \le x < \infty$ or in interval notation $[0, \infty)$.

Range: $-\infty < y < \infty$ or in interval notation $(-\infty, \infty)$.

Vertical Line Test: A graph represents a function if every vertical line you can draw intersects the graph only once (this ensures we have exactly one element $f(x)$ for each x).

3 Some Examples of Functions

3.1 Linear Functions (see Unit 4)

1. The linear equations we saw in Unit 4 are **linear functions**, such as $f(x) = 3x + 2$.

   Domain: $-\infty < x < \infty$ or in interval notation $(-\infty, \infty)$.

   Range: $-\infty < y < \infty$ or in interval notation $(-\infty, \infty)$.

2. A horizontal line $y = 2$ is a function.

   Domain: $-\infty < x < \infty$ or in interval notation $(-\infty, \infty)$.

   Range: $y \in \{2\}$ (just one number in the range, so we express it as a set).

3. A vertical line $x = 2$ is not a function.

   Domain: $x \in \{2\}$ (just one number in the domain, so we express it as a set).

   Range: $-\infty < y < \infty$ or in interval notation $(-\infty, \infty)$.

3.2 Quadratic Functions (see Unit 12)

1. We shall see in Unit 12 **quadratic functions**, such as $f(x) = 3x^2 + 5x - 2$ (polynomials of degree 2).

   Domain: $-\infty < x < \infty$ or in interval notation $(-\infty, \infty)$.

   Range: depends on the specific form of the quadratic, and is usually found from looking at the sketch of the function.

3.3 Rational Functions (see Unit 8)

1. The rational expressions we saw in Unit 8 are **rational functions**, such as $f(x) = \dfrac{5x - 2}{5x^2 + 8x + 1}$.

   Domain: All real numbers except where the denominator equals zero, since that would cause division by zero.

   Range: depends on the specific form of the rational function, and is usually found from looking at the sketch of the function.

3.4 Square Root Functions (see Unit 11)

1. We shall see in Unit 11 **square root functions**, such as $f(x) = \sqrt{x}$ (involved radicals).

   Domain: The quantity under the square root must be positive or zero.

   Range: depends on the specific form of the square root function, and is usually found from looking at the sketch of the function.

This is a short introduction to functional notation and functions. Learning about functions in more depth is one of the important topics that is studied in precalculus. If you can have a solid understanding of functional notation, domain, and range you will be in a good position when you see functions again in precalculus.

4 How Functions Can Help You Avoid Errors in Algebra

Two very common **algebra errors** are when a student writes

$(A+B)^2 = A^2 + B^2$ (this is not valid algebra)

$\sqrt{A+B} = \sqrt{A} + \sqrt{B}$ (this is not valid algebra)

For the first expression, we need to use the distributive property to simplify:

$$\begin{aligned}(A+B)^2 &= \boldsymbol{(A+B)}(A+B) \\ &= \boldsymbol{(A+B)} \cdot A + \boldsymbol{(A+B)} \cdot B && \text{(distributive property)} \\ &= A^2 + BA + AB + B^2 && \text{(distributive property again)} \\ &= A^2 + 2AB + B^2 && \text{(collect like terms)}\end{aligned}$$

For the second expression, we note that in general $\sqrt{A+B}$ cannot be simplified, and there are different properties we can use to move a square root around in an expression which we will see in Unit 10 (rationalizing expressions) and Unit 11 (solving equations with radicals).

Both of these expressions make sense if we understand something that proper functional evaluation is telling us.

The function of a sum is not the sum of the functions:

$$f(A+B) \neq f(A) + f(B)$$

This is what we need to keep in mind when doing proper functional evaluation, and are obvious if we put in specific values for A and B.

$$\begin{aligned}f(x) &= x^2 \\ f(\mathbf{1+3}) &= (\mathbf{1+3})^2 = (4)^2 = 16 \\ f(\mathbf{1}) &= (\mathbf{1})^2 = 1 \\ f(\mathbf{3}) &= (\mathbf{3})^2 = 9 \\ f(\mathbf{1}) + f(\mathbf{3}) &= 1 + 9 = 10\end{aligned}$$

We see that $(1+3)^2 = 16$ but $(1)^2 + (3)^2 = 10$. So $(A+B)^2 \neq A^2 + B^2$ in general.

$$\begin{aligned}f(x) &= \sqrt{x} \\ f(\mathbf{1+3}) &= \sqrt{\mathbf{1+3}} = \sqrt{4} = 2 \\ f(\mathbf{1}) &= \sqrt{\mathbf{1}} = 1 \\ f(\mathbf{3}) &= \sqrt{\mathbf{3}} = \sqrt{3} \\ f(\mathbf{1}) + f(\mathbf{3}) &= 1 + \sqrt{3} = 1 + \sqrt{3}\end{aligned}$$

We see that $\sqrt{1+3} = 2$ but $\sqrt{1} + \sqrt{3} = 1 + \sqrt{3}$. So $\sqrt{A+B} \neq \sqrt{A} + \sqrt{B}$ in general.

Remember these facts:

$$(A+B)^2 \neq A^2 + B^2$$
$$\sqrt{A+B} \neq \sqrt{A} + \sqrt{B}$$