This is about using the mathematical concept of *change of variables* (sometimes called substitution) which is a powerful concept and will be useful in the future. All solutions should begin by clearly identifying the change of variables that converts the equation into a quadratic equation.

Questions

1. Solve $x^4 - 11x^2 + 18 = 0$.

2. Solve $3x^4 = 10x^2 + 8$.

3. Solve $x^6 - 3x^3 = 0$.

4. Solve $x^8 - 6x^4 = 0$.

5. Solve $x^4 - 81 = 0$.

6. Solve $x^{2/5} + x^{1/5} - 2 = 0$.

7. Solve $x^{-2} + 3x^{-1} = 0$.

Solutions

1. Let $y = x^2$. From this substitution, it follows that $x^2 = y$ and $x^4 = y^2$.

$$\begin{aligned} x^4 - 11x^2 + 18 &= 0 \\ y^2 - 11y + 18 &= 0 \text{ Factor: two numbers whose product is 18 and sum is } -11\text{: } -9, -2. \\ (y-9)(y-2) &= 0 \\ y - 9 = 0 \text{ or } y - 2 &= 0 \\ y = 9 \text{ or } y &= 2 \text{ Now we must back-substitute using } y = x^2 \text{ and solve for } x. \\ x^2 = 9 \text{ or } x^2 &= 2 \\ x = \pm 3 \text{ or } x &= \pm\sqrt{2} \end{aligned}$$

Four solutions: $x = +3, -3, +\sqrt{2}, -\sqrt{2}$.

2. Let $y = x^2$. From this substitution, it follows that $x^2 = y$ and $x^4 = y^2$.

$$\begin{aligned} 3x^4 - 10x^2 - 8 &= 0 \\ 3y^2 - 10y - 8 &= 0 \\ y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ Quadratic Formula to solve for } y. \\ y &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(-8)}}{2(3)} \\ y &= \frac{10 \pm \sqrt{196}}{6} \\ y &= \frac{10 \pm 14}{6} \\ y &= \frac{10+14}{6} \text{ or } y = \frac{10-14}{6} \\ y &= 4 \text{ or } y = -\frac{2}{3} \\ x^2 &= 4 \text{ or } x^2 = -\frac{2}{3} \text{ Now we must back-substitute.} \\ x &= \pm 2 \text{ or } x = \pm\sqrt{-\frac{2}{3}} = \pm i\sqrt{\frac{2}{3}} \end{aligned}$$

Four solutions: $x = -2, 2, -\sqrt{2/3}, \sqrt{2/3}$.

3. Let $y = x^3$. From this substitution, it follows that $x^3 = y$ and $x^6 = y^2$.

$$\begin{aligned} x^6 - 3x^3 &= 0 \\ y^2 - 3y &= 0 \\ y(y-3) &= 0 \text{ Factor to solve for } y. \\ y = 0 \text{ or } y - 3 &= 0 \\ y = 0 \text{ or } y &= 3 \\ x^3 = 0 \text{ or } x^3 &= 3 \text{ Now we must back-substitute.} \\ x = 0 \text{ or } x &= \sqrt[3]{3} \end{aligned}$$

4. Let $y = x^4$. From this substitution, it follows that $x^4 = y$ and $x^8 = y^2$.

$$x^8 - 6x^4 = 0$$
$$y^2 - 6y = 0$$
$$y(y-6) = 0 \text{ Factor to solve for } y.$$
$$y = 0 \text{ or } y - 6 = 0$$
$$y = 0 \text{ or } y = 6$$
$$x^4 = 0 \text{ or } x^4 = 6 \text{ Now we must back-substitute.}$$
$$x = 0 \text{ or } x = \pm\sqrt[4]{6}$$

5. Let $y = x^2$. From this substitution, it follows that $x^2 = y$.

$$x^4 - 81 = 0$$
$$y^2 - 81 = 0$$
$$y = \pm 9$$
$$y = 9 \text{ or } y = -9$$
$$x^2 = 9 \text{ or } x^2 = -9 \text{ Now we must back-substitute.}$$
$$x = \pm 3 \text{ or } x = \pm\sqrt{-9} = \pm 3i$$

6. Let $y = x^{1/5}$. From this substitution, it follows that $x^{1/5} = y$ and $x^{2/5} = y^2$.

$$x^{2/5} + x^{1/5} - 2 = 0$$
$$y^2 + y - 2 = 0$$
$$(y+2)(y-1) = 0 \text{ Factor to solve for } y.$$
$$y + 2 = 0 \text{ or } y - 1 = 0$$
$$y = -2 \text{ or } y = 1$$
$$x^{1/5} = -2 \text{ or } x^{1/5} = 1 \text{ Now we must back-substitute.}$$
$$x = (-2)^5 \text{ or } x = 1^5$$
$$x = -32 \text{ or } x = 1$$

7. Let $y = x^{-1}$. From this substitution, it follows that $x^{-1} = y$ and $x^{-2} = y^2$.

$$x^{-2} + 3x^{-1} = 0$$
$$y^2 + 3y = 0$$
$$y(y+3) = 0 \text{ Factor to solve for } y.$$
$$y = 0 \text{ or } y + 3 = 0$$
$$y = 0 \text{ or } y = -3$$
$$x^{-1} = 0 \text{ or } x^{-1} = -3 \text{ Now we must back-substitute.}$$
$$x = \frac{1}{0} \text{ or } x = \frac{1}{-3} = -\frac{1}{3}$$

The quantity $\frac{1}{0}$ is not defined (division by zero). There is only one solution, $x = -1/3$.