Components of a Function

We will discuss what a function is in very general terms at first, and then go to more specific examples. This is not going to be mathematically rigorous, but it will give us a good basic understanding of how functions work.

A function takes an object from a set, does something to the object, and outputs a different object.

The components of a function are

- <u>The Domain</u> is just the set of objects that the function acts on.
- <u>The Range</u> is the set of all possible outputs from the function.
- <u>The Function</u> is the rule that explains what is done to the object from the domain.

Arrow Diagrams

Arrow diagrams are commonly used to represent functions, or operations on functions like $(f - g)(x)$. Arrow diagrams mainly help us understand how domains and ranges are involved in the function.

Here is an arrow diagram for an unspecified function f.
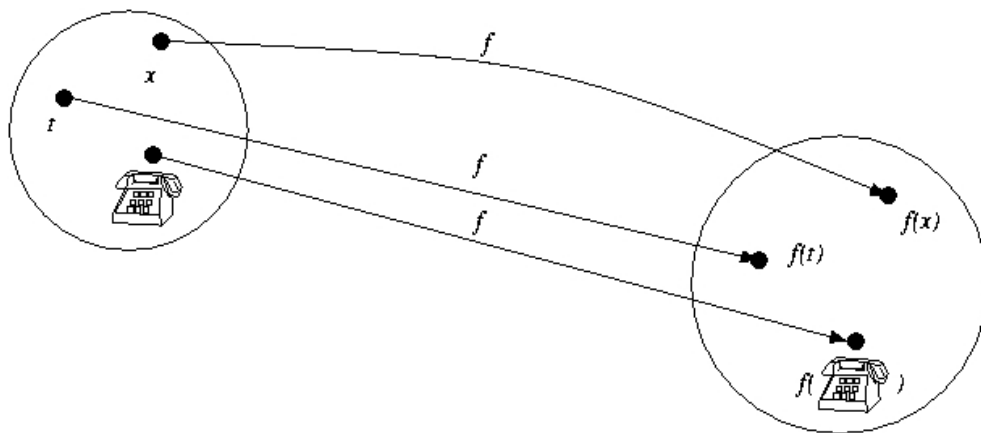
The circle on the left represents the domain. Since we can't list all possible objects in the domain, we just draw the circle and understand that is the domain. I have identified three specific objects in the domain, x, t and a telephone (I told you this wasn't going to be rigorous!)

The range is represented by the circle on the right.

The function is represented by the arrows between the domain and range. Notice that the arrows take an object from the domain, and output an object in the range.

The objects in the range are represented mathematically by $f(x)$, $f(t)$, and $f(\text{telephone})$.

We read these objects as "f of x", "f of t", and "f of telephone".

In the mathematical definition of a function, there is the word *unique*. This just means that to be function you can't have an arrow diagram that looks like the following:
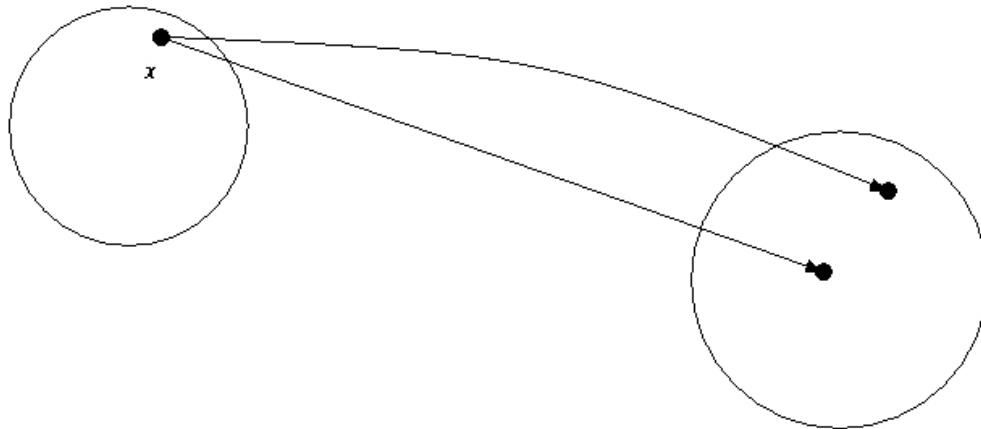
The arrow diagram above doesn't satisfy the unique requirement, so the arrows do not represent a function. They represent a relation.

A More Mathematical Example of the Arrow Diagram

Let's look at the arrow diagram for the specific function f, where f is the squaring function. Since we want to work in the real numbers, the domain is the set of all real numbers. The range is any real number that is greater than or equal to zero, since the output of the squaring function is a positive number or zero.
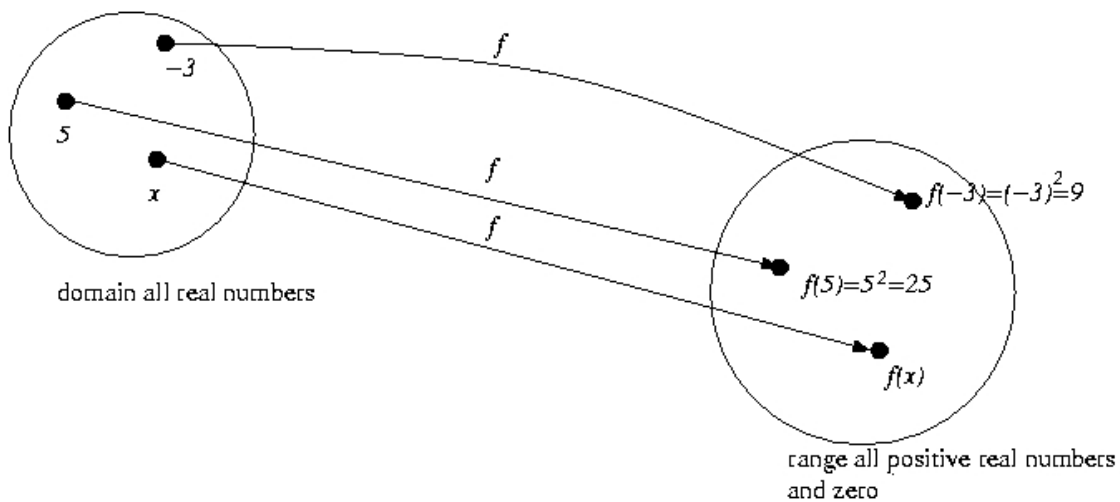
The x in the domain of the arrow diagram is meant to stand for any element in the domain–it is a *variable*. This is why we use the notation for a function $f(x)$, since the x is assumed to come from the domain and represents *any* element from the domain.

We write the domain as $x \in (-\infty, \infty)$. We write the range as $f(x) = y \in (0, \infty)$.

Examples of Proper Functional Notation

The important thing that we must be able to do is use the functional notation correctly. The functional notation is the form $f(x)$, which we read as "f of x" and that reminds us that the function is acting on the object x from the domain.

Example Given the function $f(x) = x^2$, then what is the value of the function f acting on the elements from the domain: 3, 1/3, $3x + 1$, $x + h$.

$f(3) = (3)^2$

$f(1/3) = (1/3)^2$

$f(3x + 1) = (3x + 1)^2$

$f(x + h) = (x + h)^2$

The last two can be confusing, since they involve x's. Let's think of $f(3x + 1)$ in more detail.

The quantity $3x + 1$ is an element from the domain, and you should think of it as *another element which is completely different from x*. It might help in this case to let $z = 3x + 1$, and then

$$f(3x + 1) = f(z) = z^2 = (3x + 1)^2.$$

Another way to do this is to use brackets but not fill in what goes into them right away,

$$f(\ ) = (\ )^2$$
$$f(3x + 1) = (3x + 1)^2$$

This technique applied to $f(x + h)$ would yield

$$f(\ ) = (\ )^2$$
$$f(x + h) = (x + h)^2$$

Example Given $f(x) = \dfrac{x - 1}{1 - x^2}$, find $f(x + h)$.

$f(x) = \dfrac{x - 1}{1 - x^2}$ rewrite the function

$f(\ ) = \dfrac{(\ ) - 1}{1 - (\ )^2}$ rewrite the function, this time replacing every x with brackets (do not simplify yet)

$f(x + h) = \dfrac{(x + h) - 1}{1 - (x + h)^2}$ put the quantity you want the function to operate on in all the brackets (do not simplify yet)

$f(x + h) = \dfrac{x + h - 1}{1 - (x^2 + h^2 + 2xh)} = \dfrac{x + h - 1}{1 - x^2 - h^2 - 2xh}$ now you can algebraically simplify

The key idea here is that you do not simplify anything until you have made the substitution! It is <u>incorrect</u> to write $f(x + h) = f(x) + f(h)$, so never try to simplify in this way. Remember, the algebra of functions is different than the algebra of numbers you are used to.

Example Given $f(x) = \dfrac{1}{2x^2 - x}$, simplify $\dfrac{f(x+h) - f(x)}{h}$.

Even though the question asks you to simplify the quantity, you cannot simplify it until you have made the proper functional substitutions. Let's get the quantities $f(x)$ and $f(x+h)$ before forming the quantity we are interested in.

$$
\begin{aligned}
f(x) &= \frac{1}{2x^2 - x} \\
f(\ ) &= \frac{1}{2(\ )^2 - (\ )} \\
f(x+h) &= \frac{1}{2(x+h)^2 - (x+h)} \\
&\quad \text{you can do some algebraic simplification now:} \\
&= \frac{1}{2(x^2 + h^2 + 2xh) - x - h} \\
&= \frac{1}{2x^2 + 2h^2 + 4xh - x - h}
\end{aligned}
$$

Now we have the pieces that go into the quantity we want. Since there are multiple denominators in the quantity we are interested in, let's first rewrite the division by h as multiplication by $1/h$. Then we can substitute, and start simplifying.

$$
\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left[f(x+h) - f(x)\right] \\
&\quad \text{Substitute in our results from above:} \\
&= \frac{1}{h}\left[\frac{1}{2x^2 - x} - \frac{1}{2x^2 + 2h^2 + 4xh - x - h}\right] \\
&\quad \text{algebraically simplify by getting a common denominator:} \\
&= \frac{1}{h}\left[\frac{2x^2 + 2h^2 + 4xh - x - h}{(2x^2 - x)(2x^2 + 2h^2 + 4xh - x - h)} - \frac{2x^2 - x}{(2x - x)(2x^2 + 2h^2 + 4xh - x - h)}\right] \\
&\quad \text{now use algebra to simplify as much as possible:} \\
&= \frac{1}{h}\left[\frac{(2x^2 + 2h^2 + 4xh - x - h) - (2x^2 - x)}{(2x^2 - x)(2x^2 + 2h^2 + 4xh - x - h)}\right] \\
&= \frac{1}{h}\left[\frac{2\cancel{x^2} + 2h^2 + 4xh - \cancel{x} - h - 2\cancel{x^2} + \cancel{x}}{(2x^2 - x)(2x^2 + 2h^2 + 4xh - x - h)}\right] \\
&= \frac{1}{h}\left[\frac{2h^2 + 4xh - h}{(2x^2 - x)(2x^2 + 2h^2 + 4xh - x - h)}\right] \\
&= \frac{1}{\cancel{h}}\left[\frac{\cancel{h}(2h + 4x - 1)}{(2x^2 - x)(2x^2 + 2h^2 + 4xh - x - h)}\right] \\
&= \frac{2h + 4x - 1}{(2x^2 - x)(2x^2 + 2h^2 + 4xh - x - h)}
\end{aligned}
$$

Notice that my simplification did <u>not</u> include multiplying out two polynomials in the denominator. Remember that how much to simplify and how to simplify will depend on what you want to do with the quantity you are simplifying. In the above, we usually will next want to take a limit as $h \to 0$, so the important part of the simplification is canceling the $1/h$ in front. You will learn what types of simplifications are necessary as you solve problems, and gain experience. You can always ask me if I think something can be simplified further.