Complex Rational Expressions and Rational Equations

If you have a **complex rational expression** (ie., you have multiple denominators), get a common denominator in the overall numerator and denominator, then use the rules for division to simplify. An example helps make the process clear.

$$\begin{aligned}
\frac{\frac{1-x}{x}+\frac{x}{y}}{\frac{1}{x^2y}-\frac{y}{x}} &= \frac{\frac{1-x}{x}\cdot\frac{y}{y}+\frac{x}{y}\cdot\frac{x}{x}}{\frac{1}{x^2y}-\frac{y}{x}\cdot\frac{xy}{xy}} \\
&= \frac{\frac{(1-x)y}{xy}+\frac{x^2}{xy}}{\frac{1}{x^2y}-\frac{xy^2}{x^2y}} \\
&= \frac{\left(\frac{y-xy+x^2}{xy}\right)}{\left(\frac{1-xy^2}{x^2y}\right)} \\
&= \left(\frac{y-xy+x^2}{\cancel{x}\cancel{y}}\right)\cdot\left(\frac{x^{\cancel{2}}\cancel{y}}{1-xy^2}\right) \\
&= \left(y-xy+x^2\right)\cdot\left(\frac{x}{1-xy^2}\right) \\
&= \frac{(y-xy+x^2)x}{1-xy^2}
\end{aligned}$$

When **solving equations involving rational expressions,** the following technique always works:

1. Determine the LCD of all the denominators.
2. Multiply each term in the equation by the the LCD.
3. Solve the resulting equation.
4. Check the solution–you should exclude any solution that you find which makes the LCD zero (it would result in division by zero in the original equation, so it is not allowed). These excluded solutions are called extraneous solutions.

Example Solve $\dfrac{x+11}{x^2-5x+4}+\dfrac{3}{x-1}=\dfrac{5}{x-4}$.

Factor x^2-5x+4: Need two numbers whose product is 4 and sum is -5: $-4,-1$.

$$x^2-5x+4=(x-4)(x-1).$$

The LCD for the the equation is $(x-4)(x-1)$. Multiply all terms in the equation by this LCD:

$$\begin{aligned}
&\frac{x+11}{x^2-5x+4}+\frac{3}{x-1}=\frac{5}{x-4} \\
&\frac{x+11}{\cancel{(x-4)}\cancel{(x-1)}}\cdot\cancel{(x-4)}\cancel{(x-1)}+\frac{3}{\cancel{x-1}}\cdot(x-4)\cancel{(x-1)}=\frac{5}{\cancel{x-4}}\cdot\cancel{(x-4)}(x-1) \\
&x+11+3(x-4)=5(x-1) \\
&x+11+3x-12=5x-5 \\
&4x-1=5x-5 \\
&4x-5x=-5+1 \\
&-x=-4 \\
&x=4
\end{aligned}$$

We aren't done until we verify this is actually a solution. Since $x=4$ makes the LCD zero, this is not a solution since it would result in division by zero.

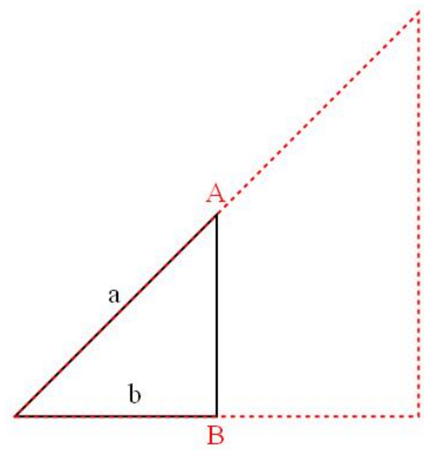
Therefore, $x=4$ is an extraneous solution (meaning it is not a solution), and the original equation has no solution.

Solving equations is obviously important, so make sure you understand the techniques for rational equations. You should be very adept at solving equations like these.

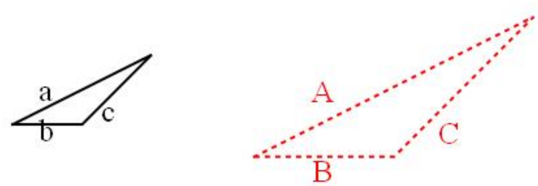
Proportion

1. Organize the information you are given.
2. Write a proportion equating the respective parts, with x as the unknown quantity. Units should cancel.
3. Solve for x.

Similar Triangles have ratios of lengths that are equal. For example:



For the above pair of similar triangles, it is true that $\frac{a}{A} = \frac{b}{B}$.



In general, we can write a variety of ratios (we could have done this for the right triangles above as well):

$$\frac{a}{A} = \frac{c}{C}, \qquad \frac{a}{A} = \frac{b}{B}, \qquad \frac{b}{B} = \frac{c}{C}.$$

We get the same ratios if we put the quantities from one triangle on the left, and the other triangle on the right:

$$\frac{a}{c} = \frac{A}{C}, \qquad \frac{a}{b} = \frac{A}{B}, \qquad \frac{b}{c} = \frac{B}{C}.$$

Proportion is something that is incredibly useful in day-to-day life. The trick to getting the ratios set up correctly is to include the units, and make sure the units cancel off. For similar triangles (where the units are the same), you can put the quantities from one triangle in the numerator, and quantities from the other triangle in the denominator. There are other ways to do this correctly, this is just my suggestion for getting the ratio correct.

Example Tim is driving his U-Haul truck, and has to hit the brakes while traveling at 55 mph due to heavy traffic. If he slows at a rate of 2 mph for every 3 seconds he is braking, how fast will be be traveling 10 seconds after he applied the brakes (assuming he brakes the entire time)?

This is a proportion problem.

$$\frac{\text{time spent braking}}{\text{reduction in speed}} \longrightarrow \frac{3\text{sec}}{2\text{mph}} = \frac{10\text{sec}}{x\text{mph}} \longleftarrow \frac{\text{given time spent braking}}{\text{unknown reduction in speed}}$$

$$\frac{3}{2} = \frac{10}{x} \text{ (notice the units cancel)}$$

$$x = 10 \cdot \frac{2}{3} = 6.67\text{mph (solve for } x\text{)}$$

Now, this is the reduction in Tim's speed, so he is traveling $55 - 6.67 = 48.33$ mph after 10 seconds.

Example It takes Rhonda 150 minutes to clean the house. It takes Pauline 100 minutes to clean the house. Working together, how long will it take them to clean the house?

In 1 minute Rhonda will finish $\frac{1}{150}$ of the cleaning.
In 1 minute Pauline will finish $\frac{1}{100}$ of the cleaning.
Let x be the time it takes them to complete the cleaning together.
In 1 minute together they will finish $\frac{1}{x}$ of the cleaning.

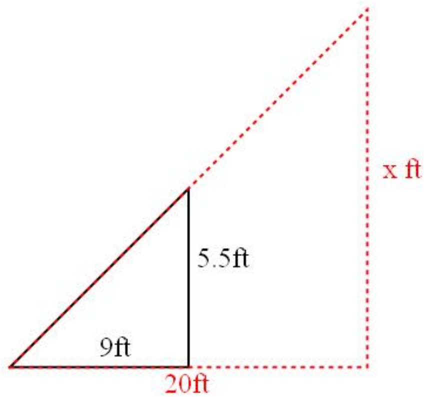
$$\begin{aligned}
\frac{1}{150} + \frac{1}{100} &= \frac{1}{x} \\
\frac{2}{300} + \frac{3}{300} &= \frac{1}{x} \\
\frac{5}{300} &= \frac{1}{x} \\
x &= 60
\end{aligned}$$

It will take 60 minutes to clean the house if they work together.

Example Simplify the following expression: $\dfrac{\frac{5}{x+4}}{\frac{1}{x-4} - \frac{2}{x^2-16}}$.

$$\begin{aligned}
\frac{\frac{5}{x+4}}{\frac{1}{x-4} - \frac{2}{x^2-16}} &= \frac{\left(\frac{5}{x+4}\right)}{\frac{1}{x-4}\cdot\frac{x+4}{x+4} - \frac{2}{(x+4)(x-4)}} \\
&= \frac{\left(\frac{5}{x+4}\right)}{\frac{x+4}{(x+4)(x-4)} - \frac{2}{(x+4)(x-4)}} \\
&= \frac{\left(\frac{5}{x+4}\right)}{\frac{x+4-2}{(x+4)(x-4)}} \\
&= \left(\frac{5}{x+4}\right) \times \frac{(x+4)(x-4)}{x+2} \\
&= \frac{5\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(x+2)} \\
&= \frac{5(x-4)}{x+2}
\end{aligned}$$

Example Sam is 5.5 ft tall and casts a shadow of 9 ft. At the same moment, the statue she is admiring in the park casts a shadow of 20 ft. How tall is the statue?



Let x be the height of the statue. Then in the diagram above the dashed red triangle represents the statue and the shadow it casts, the black triangle represents Sam and the shadow she casts.

From the diagram:

$$\begin{aligned}\frac{9\text{ft}}{20\text{ft}} &= \frac{5.5\text{ft}}{x\text{ft}}\\ x &= \frac{20}{9}(5.5) = 12.22\end{aligned}$$

So the statue is 12.22ft high.

Example In a sample survey of 145 people in Morris, it was found that 15 people of the 145 surveyed missed the Burger King that used to be here. If the population of Morris is 5,000 people, what does this survey predict is the number of people in Morris who miss the Burger King?

$$\begin{aligned}\frac{\text{Number of People in Survey}}{\text{Number of People in Survey who Miss BK}} &= \frac{\text{Number of People in Morris}}{\text{Number of People in Morris who Miss BK}}\\ \frac{145}{15} &= \frac{5000}{\text{Number of People in Morris who Miss BK}}\\ \text{Number of People in Morris who Miss BK} &= \frac{15}{145}(5000)\\ &= \frac{15}{145}(5000) = 517.2\end{aligned}$$

The survey (assuming it was a good survey, which is another question altogether) predicts that 517 people in Morris miss Burger King.