Rational Expressions and Radicals

Rules of Exponents

The rules for exponents are the same as what you saw in Section 5.1. Memorize these rules if you haven't already done so.

- $x^0 = 1$ if $x \neq 0$ (0^0 is indeterminant and is dealt with in calculus).
- Product Rule: $x^a \cdot x^b = x^{a+b}$.
- Quotient Rule: $\dfrac{x^a}{x^b} = x^{a-b}$.
- Power Rule: $(x^a)^b = x^{ab}$.
- Product Raised to Power Rule: $(xy)^a = x^a y^a$.
- Quotient Raised to a Power Rule: $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$ if $y \neq 0$.
- Negative Exponent: $x^{-a} = \dfrac{1}{x^a}$, if $x \neq 0$.

What is new in this section is the powers a and b in our rules are extended to rational numbers, so you will be working with quantities like $(8)^{1/3}$.

Radical Notation and Rules of Radicals

If x is a nonnegative real number, then $\sqrt{x} > 0$ is the principal square root of x. This is because $(\sqrt{x})^2 = x$.

Higher order roots are defined using radical notation as: $\sqrt[n]{x}$.

In words, to evaluate the expression $\sqrt[n]{x} = y$ means you are looking for a number y that when multiplied by itself n times gives you the quantity x.

$$\sqrt[4]{16} = 2 \text{ since } (2)(2)(2)(2) = 16.$$

Note that is is true that $(-2)(-2)(-2)(-2) = 16$ but we choose $+2$ since we want the principal root.

Rules of Radicals Working with radicals is important, but looking at the rules may be a bit confusing. Here are examples to help make the rules more concrete.

1. If x is a **positive real number**, then
   - $\sqrt[n]{x}$ is the nth root of x and $\left(\sqrt[n]{x}\right)^n = x$,

     $$\left(\sqrt[3]{17}\right)^3 = 17$$

     $$\left(\sqrt[3]{8}\right)^3 = 8 \text{ since } \sqrt[3]{8} = 2 \text{ since } (2)(2)(2) = 8$$

   - if n is a positive integer, we can write $x^{1/n} = \sqrt[n]{x}$.

     $$8^{1/4} = \sqrt[4]{8}$$

     $$625^{1/4} = \sqrt[4]{625} = 5 \text{ since } (5)(5)(5)(5) = 625$$

2. If x is a **negative real number**, then
   - $\left(\sqrt[n]{x}\right)^n = x$ when n is an odd integer,

     $$\left(\sqrt[3]{-6}\right)^3 = -6$$

     $$\left(\sqrt[3]{-8}\right)^3 = -8 \text{ since } \sqrt[3]{-8} = -2 \text{ since } (-2)(-2)(-2) = -8$$

- $\left(\sqrt[n]{x}\right)^n$ is not a real number when n is an even integer.

  $\left(\sqrt[2]{-6}\right)^2$ is not a real number (there is no real number that you can square and get a negative number)

3. For **all real numbers** x **(including negative values)**
   - $\sqrt[n]{x^n} = |x|$ when n is an even positive integer,

     $\sqrt[4]{(-16)^4} = |-16| = 16$ since because you take fourth power first, you are removing the negative sign

     $\sqrt[2]{(-6)^2} = \sqrt[2]{36} = \sqrt[2]{6^2} = 6$

   - $\sqrt[n]{x^n} = x$ when n is an odd positive integer.

     $\sqrt[3]{(-19)^3} = -19$

     $\sqrt[3]{(-8)^3} = \sqrt[3]{(-8)(-8)(-8)} = -8$

Product Rule for radicals: When a, b are nonnegative real numbers, $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$ (which is really just the exponent rule $(ab)^n = a^n b^n$).

Quotient Rule for radicals: When a, b are nonnegative real numbers (and $b \neq 0$), $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.

Example Evaluate $\left(\dfrac{16}{81}\right)^{3/4}$.

Solution: Since the radical for this expression would be $\left(\sqrt[4]{\dfrac{16}{81}}\right)^3$, we should look for a way to write 16/81 as (something)^4.

$$\begin{aligned}
\left(\frac{16}{81}\right)^{3/4} &= \left(\left(\frac{2}{3}\right)^4\right)^{3/4} \\
&= \left(\frac{2}{3}\right)^{4(3/4)} \\
&= \left(\frac{2}{3}\right)^{3} \\
&= \frac{2^3}{3^3} = \frac{8}{27}
\end{aligned}$$

Notice that writing this as $\left(\dfrac{16}{81}\right)^{3/4} = \sqrt[4]{\left(\dfrac{16}{81}\right)^3} = \sqrt[4]{\dfrac{4096}{531441}}$ is mathematically true, it doesn't help us simplify.

Example Simplify $\sqrt{8} + \sqrt{50} - 2\sqrt{72}$.

Since we are dealing with square roots, we simplify by looking for quantities that can be written as (something)^2.

$$\begin{aligned}
\sqrt{8} + \sqrt{50} - 2\sqrt{72} &= \sqrt{4 \cdot 2} + \sqrt{25 \cdot 2} - 2\sqrt{36 \cdot 2} \\
&= \sqrt{2^2 \cdot 2} + \sqrt{5^2 \cdot 2} - 2\sqrt{6^2 \cdot 2} \\
&= \sqrt{2^2}\sqrt{2} + \sqrt{5^2}\sqrt{2} - 2\sqrt{6^2}\sqrt{2} \\
&= 2\sqrt{2} + 5\sqrt{2} - 2 \cdot 6\sqrt{2} \\
&= -5\sqrt{2}
\end{aligned}$$

Rationalizing

Rationalizing something means getting rid of any radicals.

• To rationalize a numerator, you want to modify the expression so as to remove any radicals from the numerator.

• To rationalize a denominator, you want to modify the expression so as to remove any radicals from the denominator.

• The expression $a + \sqrt{b}$ has the conjugate expression $a - \sqrt{b}$, which can be useful when rationalizing a denominator or numerator. For example, $2 - \sqrt{43 - x}$ and $2 + \sqrt{43 - x}$ are conjugate expressions.

Example Rationalize the denominator in the expression $\dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - 2\sqrt{y}}$.

To rationalize the denominator, we multiply both the numerator and denominator by the conjugate of the denominator

$$\begin{aligned}\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - 2\sqrt{y}} &= \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - 2\sqrt{y}} \cdot \frac{\sqrt{x} + 2\sqrt{y}}{\sqrt{x} + 2\sqrt{y}} \\ &= \frac{(\sqrt{x} + \sqrt{y})(\sqrt{x} + 2\sqrt{y})}{(\sqrt{x} - 2\sqrt{y})(\sqrt{x} + 2\sqrt{y})} \\ &= \frac{x + 3\sqrt{x}\sqrt{y} + 2y}{x - 4y}\end{aligned}$$

If we wanted to, we could also rationalize the numerator by multiplying both the numerator and denominator by the conjugate of the numerator.

$$\begin{aligned}\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - 2\sqrt{y}} &= \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - 2\sqrt{y}} \cdot \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} - \sqrt{y}} \\ &= \frac{(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})}{(\sqrt{x} - 2\sqrt{y})(\sqrt{x} - \sqrt{y})} \\ &= \frac{x - y}{x - 3\sqrt{x}\sqrt{y} + 2y}\end{aligned}$$

• On page 497 the text says "If an expression contains a square root in the denominator, it is not considered simplified." I consider this statement crap. The problem is, what do we mean by *simplified*? Sometimes we may want to get rid of radicals in the denominator, but sometimes we may want to get rid of radicals in the numerator. To say one is more simplified than the other is completely misguided, in my opinion.

Back to the real question–what do we mean by simplified? Here's my answer:

> To simplify an expression may mean different things in different situations. I view something as simplified if it is in a form that makes the next thing you want to do with it easier. Do you want to sketch a function? Look for roots? Find a numeric value? Substitute it into something else? Depending on what you want to do, you may want slightly different forms.
> **Your goal is to become proficient with the algebraic techniques of simplification (rationalizing numerator, rationalizing denominator, finding a common denominator, factoring, etc), so you can easily do whatever simplification is required.**

The Square Root Function

The square root function is defined as $f(x) = \sqrt{x}$.

We have the first appearance of something called a function. The importance of the concept of a function really cannot be overstated.

We read the notation $f(x) = \sqrt{x}$ as "the function f evaluated at x is the square root of x". It is crucially important to note that $f(x)$ does not mean f times x, even though all our previous experience in this course may make us think that! This quantity is something new, and you should always read it aloud to yourself. Here are some examples:

$f(x) = \sqrt{3x+5}$ "the function f evaluated at x is the square root of 3 times x plus 5."

$f(x) = \sqrt{x-25}$ "the function f evaluated at x is the square root of x minus 25."

These are two different functions, and represent two different sets of points in the xy-plane.

Given a function, we evaluate it at different points by substituting in values for x.

The domain of a function is the set of all real numbers that we can put into the function and get a real number out.

Example You are given the function $f(x) = \sqrt{x-2}$. What is the value of this function at $x = 0$, $x = 3$ and $x = 18$? What is the domain of the function?

$$\begin{aligned} f(x) &= \sqrt{x-2} \\ f(0) &= \sqrt{0-2} = \sqrt{-2} \text{ is not a real number.} \\ f(3) &= \sqrt{3-2} = \sqrt{1} = 1 \\ f(18) &= \sqrt{18-2} = \sqrt{16} = 4 \end{aligned}$$

For the domain of the function, we note that square root function is real valued only if the quantity you are taking the square root of is nonnegative (sometimes shortened incorrectly to "you can't take the square root of a negative"), so we must have

$$\begin{aligned} x - 2 &\geq 0 \\ x &\geq 2 \end{aligned}$$

which means the domain is $x \geq 2$. We will get a real number out of our function if we put real numbers x into the function that satisfy $x \geq 2$.

Applications involving the square root

Electric Current. We can approximate the amount of current in amps I drawn by an appliance in the home using the formula

$$I = \sqrt{\frac{P}{R}}$$

where P is the power measured in watts and R is the resistance measured in ohms.

Period of a Pendulum. The period of the pendulum is the amount of time T it takes to complete one full swing back and forth. If T is measured in seconds, and the length of the pendulum is L in feet, then the period is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}.$$

A big part of precalculus is studying the behavior of functions. You will see a lot more about functions there (polynomial functions, rational functions, squaring functions, cubing functions, exponential functions, logarithmic functions, etc.).

Radicals and Factoring

You might have noticed that there is a relationship between radicals and factoring that works in some cases.

For example, what if we wanted to figure out how to simplify $y = \left(\frac{16}{81}\right)^{1/4}$? Here's something we could do.

$$
\begin{aligned}
y &= \left(\frac{16}{81}\right)^{1/4} \\
y^4 &= \left[\left(\frac{16}{81}\right)^{1/4}\right]^4 \quad \text{fourth power of both sides to remove fractional exponent} \\
y^4 &= \frac{16}{81} \quad \text{simplify} \\
81y^4 &= 16 \\
81y^4 - 16 &= 0 \quad \text{identify a difference of squares} \\
\left(9y^2\right)^2 - (4)^2 &= 0 \\
\left(9y^2+4\right)\left(9y^2-4\right) &= 0 \quad \text{first factor is prime; second is a difference of squares} \\
\left(9y^2+4\right)\left((3y)^2-2^2\right) &= 0 \\
\left(9y^2+4\right)(3y-2)(3y+2) &= 0
\end{aligned}
$$

Now the zero factor property tells us either $9y^2+4=0$ (which is not possible for y a real number), or $3y-2=0$ ($y=2/3$), or $3y+2=0$ ($y=-2/3$). We choose the positive solution since we want the principal root. So $y=2/3$.

All the above is mathematically correct, but it is obviously easier to just do the following:

$$\left(\frac{16}{81}\right)^{1/4} = \left(\left(\frac{2}{3}\right)^4\right)^{1/4} = \frac{2}{3}$$