Completing the Square

You will see this process again in precalculus and calculus, and it is definitely necessary to solve certain kinds of problems.

I want you to see it now and be able to work with it for simple cases. Here it is in general:

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c && \text{Factor so the coefficient of } x^2 \text{ is 1. Coefficient of } x \text{ is } \tfrac{b}{a} \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c && \text{the blue terms add to zero; we haven't changed the equality} \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c && \text{the red terms can be collected together as a perfect square} \\
&= a\left(\left[x + \frac{b}{2a}\right]^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left[x + \frac{b}{2a}\right]^2 - a\left(\frac{b}{2a}\right)^2 + c && \text{simplify} \\
&= a\left[x + \frac{b}{2a}\right]^2 - \frac{b^2}{4a} + c \\
&= a\left[x + \frac{b}{2a}\right]^2 - \frac{b^2 - 4ac}{4a}
\end{aligned}$$

Here it is in words:

- Factor so that there is just a 1 in front of the x^2 term.
- Identify the coefficient of the x term.
- Take half of this coefficient and square, then add and subtract so you don't change the equation.
- Fold up the perfect square you have created.
- Simplify.

This is how you derive the quadratic formula, since from here we set this equal to zero and solve for x:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
a\left[x + \frac{b}{2a}\right]^2 - \frac{b^2 - 4ac}{4a} &= 0 \\
a\left[x + \frac{b}{2a}\right]^2 &= \frac{b^2 - 4ac}{4a} \\
\left[x + \frac{b}{2a}\right]^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Right now, it looks like this doesn't do anything that the quadratic formula couldn't do, but trust me there are times when you will have to use this instead of the quadratic formula.

Example Use completing the square to solve $2x^2 + 4x + 1 = 0$.

$$2x^2 + 6x + 1 = 0 \text{ We MUST have a coefficient of 1 in front of the } x^2 \text{ before we complete the square.}$$
$$x^2 + 3x + \frac{1}{2} = 0$$
$$x^2 - 3x + \frac{1}{2} = 0 \text{ To complete the square: } \left(\frac{3}{2}\right)^2.$$
$$\underline{x^2 + 3x + \left(\frac{3}{2}\right)^2} - \left(\frac{3}{2}\right)^2 + \frac{1}{2} = 0 \text{ underlined piece is a perfect square}$$
$$\underline{\left(x + \frac{3}{2}\right)^2} = \frac{9}{4} - \frac{1}{2}$$
$$\underline{\left(x + \frac{3}{2}\right)^2} = \frac{7}{4}$$
$$x + \frac{3}{2} = \pm\sqrt{\frac{7}{4}}$$
$$x = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}$$

Quadratic Formula

Memorize: The two solutions to the equation $ax^2 + bx + c = 0$ are given by the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You can use the quadratic formula to solve quadratic equations that previously you solved by factoring.

Example Solve $\frac{1}{15} + \frac{3}{y} = \frac{4}{y+1}$.

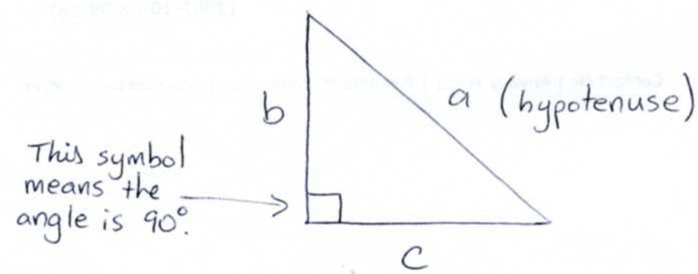
$$\frac{1}{15} + \frac{3}{y} = \frac{4}{y+1} \text{ (multiply by the LCD which is } 15y(y+1))$$
$$\frac{1}{\cancel{15}} \cdot \cancel{15}y(y+1) + \frac{3}{\cancel{y}} \cdot 15\cancel{y}(y+1) = \frac{4}{\cancel{y+1}} \cdot 15y\cancel{(y+1)} \text{ (simplify)}$$
$$y(y+1) + 45(y+1) = 60y$$
$$y^2 + y + 45y + 45 = 60y$$
$$y^2 - 14y + 45 = 0 \text{ (use quadratic formula–I always write it out)}$$
$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$y = \frac{14 \pm \sqrt{(-14)^2 - 4(1)(45)}}{2(1)}$$
$$y = \frac{14 \pm \sqrt{16}}{2} = \frac{14 \pm 4}{2} = 7 \pm 2 = 9 \text{ or } y = 5$$

Since neither $y = 9$ nor $y = 5$ makes the LCD zero, these are both solutions.

Pythagorean Theorem

The *hypotenuse* in a right triangle is the side opposite the 90 degree angle in the triangle.

Pythagorean theorem: $a^2 = b^2 + c^2$, where a is the length of the hypotenuse in a right triangle and b and c are the lengths of the other two sides.



Example A right triangle has one side of length 4cm and hypotenuse of length 5cm. What is the length of the other side?

$$\begin{aligned} a^2 &= b^2 + c^2 \\ (5)^2 &= (4)^2 + c^2 \\ 9 &= c^2 \Rightarrow c = 3 \text{ choose positive root} \end{aligned}$$

The other side has length 3cm.

Sketching Quadratics

For a quadratic function $y = f(x) = ax^2 + bx + c$, we can create a sketch by determining four things:

1. **x-intercepts:** determine these by using the quadratic formula $x_{\text{intercept}} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

   NOTE: If $b^2 - 4ac < 0$ then there are no x-intercepts (they are not real numbers).
2. **Vertex:** $(x_{\text{Vertex}}, y_{\text{Vertex}}) = \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

$$\begin{aligned} x_{\text{Vertex}} &= -\frac{b}{2a} \\ y_{\text{Vertex}} &= f\left(-\frac{b}{2a}\right) \end{aligned}$$

   How to remember this: The x-coordinate of the vertex will be right in the middle of the two x-intercepts, even if the x-intercepts are not real numbers!

$$\begin{aligned} x_{\text{Vertx}} &= \frac{1}{2}\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) \\ &= \frac{1}{2}\left(\frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}\right) \\ &= \frac{1}{2}\left(\frac{-2b}{2a}\right) \\ &= \frac{-b}{2a} \end{aligned}$$

3. **y-intercept:** evaluate at $x = 0$, so figure out what $f(0)$ is.
4. The function opens up if $a > 0$, and opens down if $a < 0$.

Example Sketch the parabola $y = 5x^2 + 4x - 12$. Label the vertex, y-intercept, and any x-intercepts on your sketch.

To get the sketch of a quadratic, we should do four things:

1. Determine if it opens up or down,
2. Determine the vertex,
3. Determine any x-intercepts (if they exist),
4. Determine the y-intercept.

A quadratic opens up if $a > 0$, and opens down if $a < 0$. Since $a = 5$ in this case, this quadratic opens up.

To get the vertex, identity $a = 5, b = 4$, and $c = -12$. Then the vertex is located at:

$$x = \frac{-b}{2a} = \frac{-4}{2(5)} = -\frac{2}{5}$$
$$y = f\left(\frac{-b}{2a}\right) = f\left(-\frac{2}{5}\right) = 5\left(-\frac{2}{5}\right)^2 + 4\left(-\frac{2}{5}\right) - 12 = -\frac{64}{5}$$

The vertex is at $\left(-\frac{2}{5}, -\frac{64}{5}\right)$.

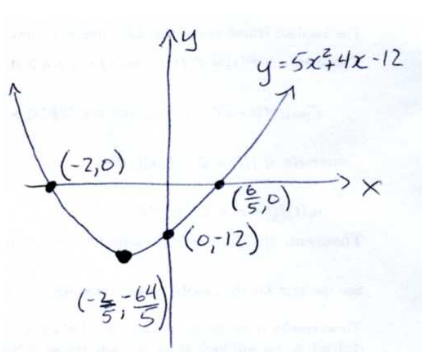
To get the x-intercepts, use the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(5)(-12)}}{2(5)} \\ &= \frac{-4 \pm \sqrt{256}}{10} \\ &= \frac{-4 \pm 16}{10} \\ &= \frac{-4 + 16}{10} \text{ or } \frac{-4 - 16}{10} \\ &= \frac{12}{10} \text{ or } \frac{-20}{10} \\ &= \frac{6}{5} \text{ or } -2 \end{aligned}$$

To get the y-intercept, evaluate $f(0)$:

$$y = f(0) = 5(0)^2 + 4(0) - 12 = -12$$

You can now put this all together to get the sketch:



Example What is the domain of the function $f(x) = \sqrt{x^2 - 3x + 1}$? Sketch your answer on a number line.

Domain: what real numbers can I put into this expression and get a real number out?

$$x^2 - 3x + 1 \geq 0$$

We only know how to algebraically solve linear inequalities, but we can sketch $y = x^2 - 3x + 1$ and get the answer from our sketch by figuring out where $x^2 - 3x + 1$ is positive.

Since $a = 1 > 0$, this quadratic opens up.

To get the vertex, identity $a = 1, b = -3$, and $c = 1$. Then the vertex is located at:

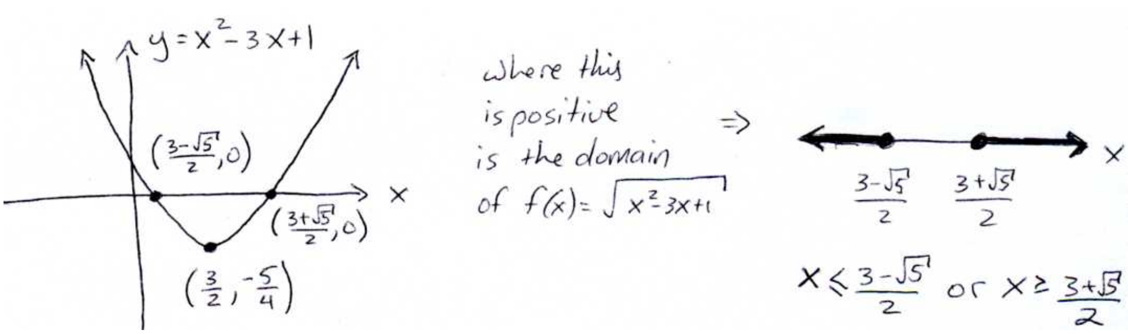
$$x = \frac{-b}{2a} = \frac{-(-3)}{2(1)} = \frac{3}{2}$$
$$y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 1 = \frac{9}{4} - \frac{9}{2} + 1 = \frac{9}{4} - \frac{18}{4} + \frac{4}{4} = \frac{-5}{4}$$

The vertex is at $\left(\frac{3}{2}, \frac{-5}{4}\right)$.

To get the x-intercepts, use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2}$$
$$= \frac{3 + \sqrt{5}}{2} \text{ or } \frac{3 - \sqrt{5}}{2}$$

We don't need the y-intercept to answer the question. You can now put this all together to get the sketch:



Transforming into a Quadratic

This is about using the mathematical concept of change of variables (sometimes called substitution) which is a powerful concept and will be useful in the future. All solutions should begin by clearly identifying the change of variables that converts the equation into a quadratic equation.

Example Solve $(x^2 + 2x)^2 - (x^2 + 2x) - 12 = 0$.

Identify the substitution $y = x^2 + 2x$. The equation then becomes

$$y^2 - y - 12 = 0$$
$$(y + 3)(y - 4) = 0 \quad \text{factor}$$
$$y + 3 = 0 \text{ or } y - 4 = 0 \quad \text{zero factor property}$$
$$y = -3 \text{ or } y = 4 \quad \text{zero factor property}$$

For each of these we get a quadratic in x to solve:

$$y = x^2 + 2x = -3$$
$$x^2 + 2x + 3 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{quadratic formula}$$
$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{-2 \pm \sqrt{-8}}{2}$$
$$x = \frac{-2 \pm i\sqrt{8}}{2}$$
$$x = \frac{-2 \pm i2\sqrt{2}}{2}$$
$$x = -1 \pm i\sqrt{2}$$

$$y = x^2 + 2x = 4$$
$$x^2 + 2x - 4 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{quadratic formula}$$
$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-4)}}{2(1)}$$
$$x = \frac{-2 \pm \sqrt{20}}{2}$$
$$x = \frac{-2 \pm 2\sqrt{5}}{2}$$
$$x = -1 \pm \sqrt{5}$$

There are four solutions to the original equation.

Two are complex valued ($x = -1 \pm i\sqrt{2}$), two are real valued ($x = -1 \pm \sqrt{5}$).

Solving power equations

This technique sometimes works, so it is useful to know.

1. Algebraically isolate the power portion on one side of equal sign.
2. Use an appropriate radical on each side of the equation to remove the power. Include $\pm\sqrt{}$ in square roots.
3. Solve the equation.
4. Check all solutions, and exclude any that do not satisfy the original equation. These excluded solutions are called extraneous solutions.

An alternate (and sometimes much easier) technique is to expand the power terms, and then collect like terms and try to factor the new expression to find the solutions. If the powers are squares, you can always do this with the aid of the quadratic formula.

Example Solve $(2x+7)^2 - \frac{3}{4} = 14$.

$$\begin{aligned}
(2x+7)^2 - \frac{3}{4} &= 14 \\
(2x+7)^2 &= 14 + \frac{3}{4} \\
(2x+7)^2 &= \frac{59}{4} \\
\sqrt{(2x+7)^2} &= \pm\sqrt{\frac{59}{4}} \\
2x+7 &= \pm\frac{\sqrt{59}}{2} \\
2x &= -7 \pm \frac{\sqrt{59}}{2} \\
x &= -\frac{7}{2} \pm \frac{\sqrt{59}}{4}
\end{aligned}$$

You have to check if these are solutions by substituting back:

$$\begin{aligned}
(2x+7)^2 - \frac{3}{4} &= \left(2\left(-\frac{7}{2} + \frac{\sqrt{59}}{4}\right) + 7\right)^2 - \frac{3}{4} \\
&= \left(-7 + \frac{\sqrt{59}}{4} + 7\right)^2 - \frac{3}{4} \\
&= \left(\frac{\sqrt{59}}{2}\right)^2 - \frac{3}{4} \\
&= \frac{59}{4} - \frac{3}{4} = \frac{56}{4} = 14
\end{aligned}$$

$$\begin{aligned}
(2x+7)^2 - \frac{3}{4} &= \left(2\left(-\frac{7}{2} - \frac{\sqrt{59}}{4}\right) + 7\right)^2 - \frac{3}{4} \\
&= \left(-7 - \frac{\sqrt{59}}{4} + 7\right)^2 - \frac{3}{4} \\
&= \left(-\frac{\sqrt{59}}{2}\right)^2 - \frac{3}{4} \\
&= \frac{59}{4} - \frac{3}{4} = \frac{56}{4} = 14
\end{aligned}$$

So both are solutions. $x = -\frac{7}{2} \pm \frac{\sqrt{59}}{4}$.