This unit did not fit into the Math 901 course, but it is an important unit to have seen before you begin precalculus.

You should memorize the rules of exponents and rules of logarithms. Practice them and keep practicing them, since they will come up often in future courses. These rules are how you correctly work with exponents when you need to algebraically manipulate them to solve equations.

Exponents & Exponent Rules

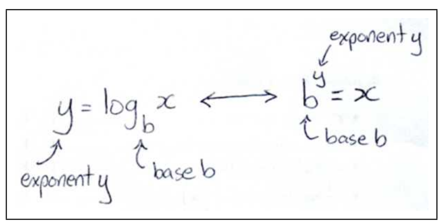
Rules of Exponents:

- $x^0 = 1$ if $x \neq 0$ (0^0 is indeterminant and is dealt with in calculus).
- Product Rule: $x^a \cdot x^b = x^{a+b}$.
- Quotient Rule: $\dfrac{x^a}{x^b} = x^{a-b}$.
- Power Rule: $(x^a)^b = x^{ab}$.
- Product Raised to Power Rule: $(xy)^a = x^a y^a$.
- Quotient Raised to a Power Rule: $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$ if $y \neq 0$.
- Negative Exponent: $x^{-n} = \dfrac{1}{x^n}$, if $x \neq 0$.

Logarithms & Logarithm Rules

Logarithms are another way of dealing with exponents. The following is true:

$\boxed{\text{If } y = \log_b x \text{ then it is true that } x = b^y.}$



Read $y = \log_b x$ as "y is equal to the logarithm with base b of x". The logarithm represents the exponent in $x = b^y$.

The base for the logarithm should be a positive number not equal to one ($y = \log_b(x)$ is defined only for $b > 0$ and $b \neq 1$).

The logarithm of a negative number is not a real number. Consider $y = \log_2(-1)$, which means $2^y = -1$. There is no way to pick a real number y that makes this true since 2^y is always greater than zero!

Rules of Logarithms:

- $\log_b b = 1$ if $b \neq 1$.
- $\log_b 1 = 0$ if $b \neq 1$.
- If $\log_b x = \log_b y$ then $x = y$ if $b \neq 1$.
- Product Rule: $\log_b(MN) = \log_b M + \log_b N$.
- Quotient Rule: $\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$.
- Power Rule: $\log_b(M^p) = p \log_b M$.

Logarithms and exponents are closely related–you can think of logarithms as "the opposite" of exponents (to put it more mathematically, the logarithmic function is the inverse of the exponential function).

In precalculus you will study the exponential function $y = f(x) = 2^x$ and the logarithmic function $y = f(x) = \log_2 x$ (along with bases other than 2). But that must wait until you have a stronger understanding of what *functions* really are. For now, you should be able to work with the notation $\log_b a$ to simplify expressions by using the logarithm rules to convert expressions into the equivalent exponent form, and then use your knowledge of exponents to figure out what the answer must be.

What use are logarithms? We use logarithms because they "bring exponents down", and allow us to solve more complicated mathematical models of the world.

Example A population of bacteria doubles every hour. The initial population is given by 500 bacteria. Questions we might want to answer about this situation are:

- What is the population after 2,3,4, and 5 hours?
- What is the population after t hours?
- How many hours will it take for the population to reach one million bacteria?
- What if the population tripled every hour, how would that change our answers?

Let's take each question in turn.

What is the population after 2,3,4, and 5 hours?

The population doubles every hour, so we can write a little table to answer this question.

Time (hours)	Population of Bacteria
1	500
2	1000
3	2000
4	4000
5	8000

What is the population after t hours?

This question is answered by looking at the table we created earlier and trying to spot a pattern. Whenever we are looking for a pattern, we should leave things unevaluated. With that in mind, here's the table again:

Time (hours)	Population of Bacteria
1	500
2	2×500
3	$2^2 \times 500$
4	$2^3 \times 500$
5	$2^4 \times 500$

You shouldn't be surprised to see exponents occurring! Now we look for a pattern. It appears that the population of bacteria after t hours is given by $2^t \times 500$. This is actually a pretty awesome result, since the formula

$$\text{Population } = 500 \times 2^t$$

can be used for times like $t = 50$ minutes $= 5/6$ hours, etc. Our table could not give us the population at that time!

How many hours will it take for the population to reach one million bacteria?

To answer this question we need to solve the equation:

$$500 \times 2^t = 1,000,000$$
$$2^t = 2,000 \Rightarrow t = \log_2(2,000)$$

So it will take $t = \log_2(2,000)$ hours to reach one million bacteria. You would have to use a calculator to evaluate this, since we can't guess what power 2 should be raised to to give us 2,000. A calculator will tell you this is $t = 10.96$ hours.

What if the population tripled every hour, how would that change our answers?

Answering this question is easy given our earlier analysis. It shouldn't take you long to figure out that the population would be given by

$$\text{Population } = 500 \times 3^t$$

and it wold therefore take $t = \log_3(2,000) = 6.9$ hours to reach one million.

Note: Although we phrased this example in terms of a population of bacteria, the same techniques would work for any population of animals (fish in a pond, rats in a field, etc), for how money grows in an investment, for how interest accrues on a debt, for how radioactive materials decay–and more. This is a pretty powerful idea!

Examples

I've given you some involved problems here, some of which you are likely to see again in precalculus.

Example Evaluate $\log_3\left(\frac{1}{27}\right)$.

To answer this, let y be the quantity you want to simplify and then write in the equivalent exponent form:

$$y = \log_3\left(\frac{1}{27}\right) \Rightarrow 3^y = \frac{1}{27}$$

Now, we know that $3^3 = 27$, and therefore $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$, so $y = -3$.

Example Solve for x when $2\log_{10} x + \log_{10} 5 = 1$.

You solve this by using the rules of logarithms and exponents. It can help you memorize the rules if you write the rule down every time you use it when solving a problem.

$$\begin{aligned}
2\log_{10} x + \log_{10} 5 &= 1 \text{ Use Power Rule: } p\log_b M = \log_b(M^p) \\
\log_{10}(x^2) + \log_{10} 5 &= 1 \text{ Use Product Rule: } \log_b M + \log_b N = \log_b(MN) \\
\log_{10}(5x^2) &= 1 \text{ Use Basic Relationship: If } y = \log_b x \text{ then it is true that } x = b^y \\
5x^2 &= 10^1 \text{ Now solve for } x\text{, using techniques from previous units} \\
x^2 &= 2 \\
x &= \pm\sqrt{2}
\end{aligned}$$

We must check these solutions, since you can get extraneous solutions appearing.

Check $\sqrt{2}$:

$$\begin{aligned}
&2\log_{10}\sqrt{2} + \log_{10} 5 = 1 \\
&\log_{10}(\sqrt{2}^2) + \log_{10} 5 = 1 \\
&\log_{10} 2 + \log_{10} 5 = 1 \\
&\log_{10}(2 \times 5) = 1 \\
&\log_{10}(10) = 1 \text{ True, so } x = \sqrt{x} \text{ is a solution.}
\end{aligned}$$

Check $-\sqrt{2}$:

$$2\log_{10}(-\sqrt{2}) + \log_{10} 5 = 1$$

We already have problems. The logarithm of a negative is not a real number. So $x = -\sqrt{2}$ is not a solution.

Example Solve for x if $\log_3(4x+6) - \log_3(x-1) = 2$.

$$\log_3(4x+6) - \log_3(x-1) = 2 \text{ Use Quotient Rule: } \log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$$
$$\log_3\left(\frac{4x+6}{x-1}\right) = 2 \text{ Use Use Basic Relationship: If } y = \log_b x \text{ then it is true that } x = b^y$$
$$3^2 = \frac{4x+6}{x-1} \text{ Now solve for } x\text{, using techniques from previous units}$$
$$9(x-1) = 4x+6$$
$$9x-9 = 4x+6$$
$$9x-4x = 6+9$$
$$5x = 15 \Rightarrow x = 3$$

Check:

$$\log_3(4(3)+6) - \log_3(3-1) = 2$$
$$\log_3(18) - \log_3(2) = 2$$
$$\log_3\left(\frac{18}{2}\right) = 2$$
$\log_3 9 = 2 \Rightarrow 3^2 = 9$ which is True, so $x = 3$ is a solution.

Example Solve for x if $\log_{10} x + \log_{10}(2x+1) = 1$.

$$\log_{10} x + \log_{10}(2x+1) = 1 \text{ Use Product Rule: } \log_b M + \log_b N = \log_b(MN)$$
$$\log_{10}(x(2x+1)) = 1 \text{ Use Use Basic Relationship: If } y = \log_b x \text{ then it is true that } x = b^y$$
$$10^1 = x(2x+1) \text{ Now solve for } x\text{, using techniques from previous units}$$
$$10 = 2x^2 + x$$
$$2x^2 + x - 10 = 0 \text{ use quadratic formula}$$
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \text{ where } a = 2, b = 1, c = -10$$
$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(2)(-10)}}{2(2)}$$
$$x = \frac{-1 \pm \sqrt{81}}{4}$$
$$x = \frac{-1 \pm 9}{4}$$
$$x = \frac{-1-9}{4} \text{ or } \frac{-1+9}{4}$$
$$x = \frac{-5}{2} \text{ or } 2$$

We exclude the negative solution, since it would lead to $\log_{10}(-5/2)$ which is not a real number.

Check $x = 2$:

$$\log_{10} 2 + \log_{10}(2(2)+1) = 1$$
$$\log_{10} 2 + \log_{10} 5 = 1$$
$$\log_{10}(2 \times 5) = 1$$
$\log_{10}(10) = 1 \Rightarrow 10^1 = 10$ True! So $x = 2$ is a solution.

The Natural Logarithm

There is a special base for exponents that is the preferred base in mathematics. It is the irrational number (a nonterminating, nonrepeating decimal) $e = 2.7182818284590452354\ldots$. You will learn why this base is preferred in calculus. So the exponential function is e^x, which you will study in precalculus.

The logarithm associated with base e is called the *natural logarithm*, denoted $\ln x$. (it is "el-n" instead of "n-el" since historically it was first called *logarithmus naturalis*).

Although e is a number, the natural logarithm is not a number, it is an operator–so it must always be acting on something. This is similar to other operators like sine in trigonometry.

Rules of Exponents for base e:

- $e^0 = 1$ if $x \neq 0$ (0^0 is indeterminant and is dealt with in calculus).
- Product Rule: $e^a \cdot e^b = e^{a+b}$.
- Quotient Rule: $\dfrac{e^a}{e^b} = e^{a-b}$.
- Power Rule: $(e^a)^b = e^{ab}$.
- Negative Exponent: $e^{-n} = \dfrac{1}{e^n}$.

Rules of Logarithms for natural logarithms:

- $\ln e = 1$ if $b \neq 1$.
- $\ln 1 = 0$ if $b \neq 1$.
- If $\ln x = \ln y$ then $x = y$.
- Product Rule: $\ln(MN) = \ln M + \ln N$.
- Quotient Rule: $\ln\left(\dfrac{M}{N}\right) = \ln M - \ln N$.
- Power Rule: $\ln(M^p) = p \ln M$.

You will spend a lot of time learning about exponential and logarithm functions in precalculus, so think of this unit as an introduction to the notation and a chance to practice the rules associated with exponents and logarithms.