Fractions

(see the Chapter Organizer on page 53 of the text for more details of each procedure)

Mathematics $\longrightarrow$ Fractions are preferred. For applied problems convert to decimal as the last step in your solution.

Science Courses $\longrightarrow$ Fractions for theoretical problems. For applied problems decimals with error analysis.

In mathematics it is usually far preferable to work with fractions rather than decimals. You must be very comfortable using fractions.

The arithmetic of decimals are discussed in Section 0.4, which I am not having you do. Most of you will simply use a calculator if you need to do decimal arithmetic in the future.

Prime Factorization

Technique: Find a number that divides evenly, and continue the process until you have a prime factorization:

$$\begin{aligned} 168 &= 2 \times 84 \\ &= 2 \times 2 \times 42 \\ &= 2 \times 2 \times 2 \times 21 \\ &= 2 \times 2 \times 2 \times 3 \times 7 \\ &= 2^3 \times \times 3^1 \times 7^1 \end{aligned}$$

Simplifying Fractions

Technique: Prime factor numerator and denominator, then **cancel common factors**.

$$\frac{42}{140} = \frac{2 \times 3 \times 7}{2 \times 2 \times 7 \times 5} = \frac{\cancel{2} \times 3 \times \cancel{7}}{\cancel{2} \times 2 \times \cancel{7} \times 5} = \frac{3}{2 \times 5} = \frac{3}{10}$$

• Changing fractions to equivalent fractions with a given denominator

$\frac{23}{3} = \frac{?}{9}$ since $3 \times 3 = 9$ in the denominators, multiply numerator and denominator by 3, $\frac{23 \times 3}{3 \times 3} = \frac{69}{9}$.

You need to have common denominators before you can add or subtract fractions.

$$\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d} \qquad \frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}$$

Finding the LCD (least common denominator) of two or more fractions

Technique: Look for the prime factorization of the two denominators. Collect common factors.

Example Find LCD of $\frac{3}{16}$ and $\frac{7}{28}$.

Prime factorization of the denominators:

$$\begin{aligned} 16 &= 2 \times 2 \times 2 \times 2 \\ 28 &= 2 \times 2 \qquad \times 7 \\ \hline \text{LCD} &= 2 \times 2 \times 2 \times 2 \times 7 = 112 \end{aligned}$$

You can now use this to add or subtract the fractions:

$$\begin{aligned}\frac{3}{16}+\frac{7}{28} &= \frac{3\times 7}{16\times 7}+\frac{7\times 4}{28\times 4} \text{ notice what you multiply by are the bits missing in the LCD table!}\\ &= \frac{21}{112}+\frac{28}{112} \text{ now that you have common denominators, you can add the fractions}\\ &= \frac{21+28}{112}\\ &= \frac{49}{112} \text{ now you can prime factor the numerator and denominator and simplify the fraction}\\ &= \frac{\cancel{7}\times 7}{\cancel{7}\times 16}\\ &= \frac{7}{16} \text{ done!}\end{aligned}$$

Example Simplify $\frac{1}{5}+\frac{8}{25}-\frac{9}{21}$.

Prime factorization of the denominators:

$$\begin{aligned}5 &= 5\\ 25 &= 5\times 5\\ 21 &= \quad 3\times 7\\ \hline \text{LCD} &= 5\times 5\times 3\times 7 = 525\end{aligned}$$

You can now use this to add or subtract the fractions:

$$\begin{aligned}\frac{1}{5}+\frac{8}{25}-\frac{9}{21} &= \frac{1\times 5\times 3\times 7}{5\times 5\times 3\times 7}+\frac{8\times 3\times 7}{25\times 3\times 7}-\frac{9\times 5\times 5}{21\times 5\times 5}\\ &= \frac{105}{525}+\frac{168}{525}-\frac{225}{525}\\ &= \frac{105+168-225}{525}\\ &= \frac{48}{525}\\ &= \frac{\cancel{3}\times 16}{\cancel{3}\times 175}\\ &= \frac{16}{175}\end{aligned}$$

You can add fractions using a denominator that isn't the least common denominator.

$$\frac{1}{4}+\frac{3}{8}=\frac{1\times 8}{4\times 8}+\frac{3\times 4}{8\times 4}=\frac{8}{32}+\frac{12}{32}=\frac{8+12}{32}=\frac{20}{32}=\frac{\cancel{4}\times 5}{\cancel{4}\times 8}=\frac{5}{8}$$

- Multiplying fractions: Does NOT require an LCD! $\frac{23}{44}\times\frac{22}{13}=\frac{23\times 22}{44\times 13}=\frac{23\times\cancel{11}\times\cancel{2}}{\cancel{2}\times 2\times\cancel{11}\times 13}=\frac{23}{2\times 13}=\frac{23}{26}$.
- Dividing fractions: Does NOT require an LCD! $\frac{3}{55}\div\frac{2}{5}=\frac{3}{55}\times\frac{5}{2}=\frac{3\times 5}{55\times 2}=\frac{3\times\cancel{5}}{\cancel{5}\times 11\times 2}=\frac{3}{11\times 2}=\frac{3}{22}$.

To ADD or SUBTRACT fractions you MUST use a common denominator.

To MULTIPLY fractions does NOT require an LCD, simply multiply numerators and multiply denominators.

To DIVIDE fractions does NOT require an LCD. You invert the fraction you are dividing by and multiply.

Advice: I highly recommend that you get in the habit of using brackets when you have multiple denominators (the text does not do this, and I think that is a problem).

$\frac{1}{4} \div \frac{4}{5}$ is written as $\frac{\left(\frac{1}{4}\right)}{\left(\frac{4}{5}\right)}$ instead of $\frac{\frac{1}{4}}{\frac{4}{5}}$.

You will avoid many errors if you do this.

Changing improper fractions to mixed numbers

$\frac{25}{6} = 4\frac{1}{6}$ since 6 goes into 25 4 times with a remainder of 1.

Improper fractions are used more frequently in higher math than mixed numbers. It is far more common to see $\frac{4}{3}$ instead of $1\frac{1}{3}$. Part of the reason for this is that $1\frac{1}{3} = 1 + \frac{1}{3}$ but $ab = a \times b$, which can be confusing.

Bottom line–get comfortable with improper fractions.

Changing mixed numbers to improper fractions

$$4\frac{1}{6} = 4 + \frac{1}{6} = \frac{4}{1} + \frac{1}{6} = \frac{4 \times 6}{1 \times 6} + \frac{1}{6} = \frac{4 \times 6}{6} + \frac{1}{6} = \frac{4 \times 6 + 1}{6} = \frac{25}{6}.$$

Adding and subtracting mixed numbers: change to improper fraction

Multiplying and dividing mixed numbers: change to improper fraction

Percents

The percent sign % can be thought of as "per one hundred", or $\% \leftrightarrow \frac{1}{100}$.

• Changing a percent to a decimal

$$6.7\% = 6.7 \times \frac{1}{100} = \frac{6.7}{100} = 0.067.$$

Advice: Get in the habit of writing the leading zero when using decimals (0.067 instead of .067) since that makes the number clearer to read and helps avoid errors.

• Changing a decimal to a percent

$0.00078 = 0.078\%$ (move decimal two places to the left)

• Finding a percent of a number

27.5% of 5.1 is found by the following:

$$27.5\% \times 5.1 = 0.275 \times 5.1 = 1.4025$$

This makes sense since the answer should be close to 25% of 5 which is 1.25.

• Finding what percent of one number is of another number

What percent of 5 is 160?

$$\frac{160}{5} = 32 = 3200\%.$$

This makes sense since 160 is larger than 5, so it should be greater than 100% of 5.

Barry's Comments

As your knowledge of mathematics progresses, you will learn different ways of doing things that may be easier. For example, to change fractions to a new denominator, you can think of the problem as one in which you solve for an unknown, which is discussed in Unit 3 Equations and Inequalities.

$\frac{23}{3} = \frac{x}{9}$ solve for x
$\frac{23}{3} \times 9 = \frac{x}{9} \times 9$ multiply both sides of equation by 9
$\frac{23 \times 3 \times \cancel{3}}{\cancel{3}} = \frac{x}{\cancel{9}} \times \cancel{9}$ simplify
$69 = x$
so $\frac{23}{3} = \frac{69}{9}$

Examples

Example 0.2.78 Carl bought a 20 gallon aquarium. He put $17\frac{3}{4}$ gallons of water into the aquarium, but it looked too low so he added $1\frac{1}{4}$ gallons of water. Then he added plants and gravel, but the water was too high so he removed $2\frac{2}{3}$ gallons of water. How many gallons of water are now in the aquarium?

$$17\frac{3}{4} + 1\frac{1}{4} - 2\frac{2}{3} = \frac{71}{4} + \frac{5}{4} - \frac{8}{3}.$$

The LCD for these improper fractions is 12.

$$\frac{71}{4} + \frac{5}{4} - \frac{8}{3} = \frac{71 \times 3}{4 \times 3} + \frac{5 \times 3}{4 \times 3} - \frac{8 \times 4}{3 \times 4} = \frac{213}{12} + \frac{15}{12} - \frac{32}{12} = \frac{213 + 15 - 32}{12} = \frac{196}{12} = 16\frac{1}{3}$$

Since 12 divides into 196 16 times with a remainder of $\frac{4}{12} = \frac{1}{3}$, there is $16\frac{1}{3}$ gallons of water in the tank.

Example 0.2.81 A country club maintains the putting greens with a grass height of $\frac{7}{8}$ inches. The grass on the fairways is maintained at $2\frac{1}{2}$ inches. How much must the mower blade be lowered if the person mowing the fairways uses the same machine to mow the greens?

$$2\frac{1}{2} - \frac{7}{8} = \frac{5}{2} - \frac{7}{8} = \frac{5 \times 4}{2 \times 4} - \frac{7}{8} = \frac{20}{8} - \frac{7}{8} = \frac{20 - 7}{8} = \frac{13}{8} = 1\frac{5}{8}$$

The blade must be lowered $1\frac{5}{8}$ inches.

Example (based on 0.3) A denim shirt requires $2\frac{3}{4}$ yards of material to make. If you have 142 yards of material, how many shirts can you make (assume there is no wasted material).

$$\frac{142}{\left(2\frac{3}{4}\right)} = \frac{142}{\left(\frac{11}{4}\right)} = \frac{142}{1} \times \frac{4}{11} = \frac{568}{11} \sim 51.6$$

You only have enough material to make 51 shirts. Notice in this problem it made sense to convert to a decimal at the end.

Example 0.5.31 Dave took a multiple choice exam with 80 questions and answered 68 of them correctly. What was his percentage grade for the exam?

$$\frac{68}{80} = 0.85 = 85\%.$$

Example 0.5.34 Music cds have a failure rate of 1.8% (meaning 1.8% of them are defective). If 36,000 cds were manufactured in a week, how many of them are defective?

$$1.8\% \times 36,000 = 0.018 \times 36,000 = 648 \text{ were defective.}$$

Example What percent by weight of carbon tetrachloride, CCl_4, is carbon, C? The atomic weights of these elements are (C, 12: Cl, 35).

Carbon tetrachloride is made up of 1 carbon and 4 chlorine.

The atomic weight of carbon tetrachloride is therefore $1 \times 12 + 4 \times 35 = 152$ atomic units.

The fraction of this weight that is carbon is $\frac{1 \times 12}{152} = \frac{3 \times \cancel{4}}{38 \times \cancel{4}} = \frac{3}{38}$.

Since $\frac{3}{38} \sim 0.07894 = 7.89\%$, carbon tetrachloride gets roughly 7.89% of its weight from carbon.

It made sense to convert to decimals at the end.

Example 0.5.64 Dave Bagley traveled 24,500 miles last year. He is a salesperson and 74% of his mileage is business related. He was planning to deduct 31 cents per business mile on his income tax return, but his accountant told him he can deduct 35 cents per business mile. How much would his deduction increase if he uses the new larger amount?

First, we need to determine how many miles are business related miles.

$$24,500 \text{ miles} \times 74\% \text{ business related} = 24,500 \times 74\% \text{ business related miles}$$

Let's drop the units and figure this out:

$$24,500 \times 74\% = 24,500 \times 0.74 = 18,130.$$

So Dave has 18,130 business miles.

His deduction at 31 cents per mile (or \$0.31 per mile) would be:

$$18,130 \times \$0.31 = \$5620.30.$$

His deduction at 35 cents per mile (or \$0.35 per mile) would be:

$$18,130 \times \$0.35 = \$6345.50.$$

Dave's accountant increased his deduction by

$$\$6345.50 - \$5620.30 = \$725.20.$$