Equations and Inequalities

• An equation involves an equal sign and indicates that two expressions have the same value.

$x + 42 = 67(4 - x)$ is an equation, and means $x + 42$ has the same value as $67(4 - x)$.

• Equivalent equations are equations that have *exactly* the same solution.

Solving an equation typically involves using the rules of algebra to construct a series of equivalent equations until you determine a numerical solution for an unknown variable.

The Addition Principle: If the same number is added to both sides of an equation, the results on both sides are equal in value (you have constructed an equivalent equation).

$x + 42 = 67(4 - x)$ is an equation,
$x + 42 + 76 = 67(4 - x) + 76$ is an equivalent equation.

The Multiplication Principle: If both sides of an equation are multiplied by the same nonzero number, the results on both sides are equal in value (you have constructed an equivalent equation).

$x + 42 = 67(4 - x)$ is an equation,
$132(x + 42) = 132 \times 67(4 - x)$ is an equivalent equation.

The Division Principle: If both sides of an equation are divided by the same nonzero number, the results on both sides are equal in value (you have constructed an equivalent equation).

$x + 42 = 67(4 - x)$ is an equation,
$\frac{(x + 42)}{69} = \frac{67(4 - x)}{69}$ is an equivalent equation.

Note that you have to be careful with division and multiplication. Make sure you multiply or divide each entire side of the equation,

$x + 42 = 1 - x$ is an equation,
$132 \times x + 42 = 132 \times 1 - x$ is not an equivalent equation. This is like driving on the wrong side of the road.
$x + 42 \div 69 = 1 - x \div 69$ is not an equivalent equation. This is like driving on the wrong side of the road.

The problem with the above is that you are not multiplying or dividing all of the terms on the each side, and thus you have changed the equation. Use parentheses and you will be ok:

$x + 42 = 1 - x$ is an equation,
$132 \times (x + 42) = 132(1 - x)$ is an equivalent equation. This is like obeying the traffic laws.
$(x + 42) \div 69 = (1 - x) \div 69$ is an equivalent equation. This is like obeying the traffic laws.

Solving an equation of the form $ax + b = cx + d$ (or even slightly more complicated equations) involves constructing a series of equivalent equations that ends with the equivalent equation $x =$ a number. The following steps are required:

1. Clear any parentheses, and simply as much as possible (simplifying is just advice to make things easier).
2. Collect like terms using the Addition Principle if necessary.
3. Isolate the variable term.
4. Use the Division Principle to isolate the variable.
5. Check your answer by substituting back in the original equation to see if your answer is correct.

The process of solution of an inequality is the same as for an equation, except that the inequality is reversed if you multiply or divide by a negative number.

• When sketching an inequality you use an open circle if the endpoint is not included, and a filled in circle if the endpoint is included. Here's how you can remember this:

For $<$ (one thing)	draw ○	one thing (draw the circle)
For $>$ (one thing)	draw ○	one thing (draw the circle)
For $\geq$ (two things)	draw ●	two things (draw the circle and then shade it in)
For $\leq$ (two things)	draw ●	two things (draw the circle and then shade it in)

Interval Notation and Set Notation for Inequalities

$a \leq x \leq b$ is equivalent to $x \epsilon [a, b]$

$a < x < b$ is equivalent to $x \epsilon (a, b)$

$a \leq x < b$ is equivalent to $x \epsilon [a, b)$

$a < x \leq b$ is equivalent to $x \epsilon (a, b]$

Literal Equations have many unspecified variables, but you solve them using the same techniques. You can just can't simplify as much since you are working with variables instead of numbers.

The Combined Gas Law A nice example of a literal equation used in chemistry is the Combined Gas Law, which states that for a gas under two different sets of conditions (labeled by the subscript 1 or 2), it is true that

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}.$$

Example An ideal gas in state 1 has $P_1 = 3$ Pa, $V_1 = 20$ cm^3, and $T_1 = 40$ K. This gas is then adjusted so the pressure is $P_2 = 4$ Pa and the volume is $V_2 = 50$ cm^3. What is the temperature of the gas in state 2, using the Combined Gas Law?

$$\begin{aligned}
\frac{P_1V_1}{T_1} &= \frac{P_2V_2}{T_2} \text{ write the equation you will start with} \\
\frac{(3\text{Pa})(20\text{cm}^3)}{(40\text{K})} &= \frac{(4\text{Pa})(50\text{cm}^3)}{T_2} \text{ substitute in the values} \\
T_2 &= \frac{(4\text{Pa})(50\text{cm}^3)}{(3\text{Pa})(20\text{cm}^3)}(40\text{K}) \text{ solve for } T_2 \\
T_2 &= \frac{(4\cancel{\text{Pa}})(50\cancel{\text{cm}^3})}{(3\cancel{\text{Pa}})(20\cancel{\text{cm}^3})}(40\text{K}) \text{ solve for } T_2\text{, cancel units} \\
T_2 &= \frac{(4)(50)}{(3)(20)}(40)\text{K} = \frac{400}{3}\text{K} \sim 133\text{K}
\end{aligned}$$

I split the canceling of units into it's own step, but it need not be. Show as much detail as you need to get the simplification done correctly.

Example Solve the Combined Gas Law for T_2.

$$\begin{aligned}
\frac{P_1V_1}{T_1} \times T_2 &= \frac{P_2V_2}{\cancel{T_2}} \times \cancel{T_2} \\
\frac{\cancel{T_1}}{\cancel{P_1V_1}} \times \frac{\cancel{P_1V_1}}{\cancel{T_1}} \times T_2 &= \frac{T_1}{P_1V_1} \times P_2V_2 \\
T_2 &= \frac{T_1P_2V_2}{P_1V_1}
\end{aligned}$$

Triangle facts

1. The sum of the interior angles of a triangle is 180 degrees.
2. An *equilateral triangle* has three sides of equal lengths and three angles that measure 60 degrees.
3. An *isosceles triangle* has two sides of equal length. The two angles opposite the equal sides are also equal.
4. A *right triangle* has one angle that is 90 degrees.

• The Volume and Surface Area formulas are certainly important, but I will give them to you if you need them. Other classes you take may require you to have them memorized.

Word Problems

In Section 2.6, the problems become more complicated and a couple of techniques are introduced to help you organize your solution. The first is on pg 160:

1. **Understand the problem.**
   Here you read the problem, draw a sketch if it will help, organize any given information and try to figure out what it is you asked to find.
2. **Write an equation.**
   Here you write down any equations that might help (area of circle, perimeter of a parallelogram, etc), and label any values with quantities which are given.
3. **Solve and state the answer.**
   Once you have determined an equation that contains the unknown quantity, solve for the unknown quantity.
4. **Check.**
   If you can, check your answer somehow.

If the problem is particularly complicated, you might want to use something like the *Mathematical Blueprint for Problem Solving* to help you understand the problem. Here, you have headings which you fill in with given information until a method of solution presents itself. I use these in the homework.

Here is a simple example of how you can fill the Blueprint out:

Example A Motorcycle shop maintains an inventory of four times as many new bikes as used bikes. If they have space for sixty bikes total, how many new bikes and how many used bikes do they have in stock?

Gather Facts	Assign Variables	Basic Formula or Equation	Key Points
• There are 60 bikes total.	• There are x used bikes. • There are four times as many new bikes as used bikes, so there are $4x$ new bikes. • The total number of bikes is the sum of the new and used bikes, $4x + x$.	• $4x + x = 60$.	The number of bikes (both new and used) should work out to be an integer.

Solve the equation:

$$4x + x = 60 \Rightarrow x = 12.$$

So there are 12 used bikes, and $4 \times 12 = 48$ used bikes in the shop.

Check: $12 + 4(12) = 12 + 48 = 60$. ✓

This is not meant to be overly cumbersome–it is meant to help you organize your thoughts. Sometimes I leave columns blank, and sometimes I don't use the Blueprint at all. Use it if it helps you, but it is not necessary to use it to create a correct solution.

Examples

Example 2.4.34 Solve $\frac{2}{3}(x+4) = 6 - \frac{1}{4}(3x-2) - 1$.

Remember, you might choose a different route to the solution that is entirely correct.

The goal is first to isolate a single term with x in it on one side of the equation.

$$\begin{aligned}
\frac{2}{3}(x+4) &= 6 - \frac{1}{4}(3x-2) - 1 \\
\frac{2}{3}x + \frac{8}{3} &= 6 - \frac{3}{4}x + \frac{2}{4} - 1 \text{ (I choose to clear parentheses first)} \\
\frac{2}{3}x + \frac{8}{3} &= {\color{blue}6} - \frac{3}{4}x {\color{blue}+ \frac{2}{4} - 1} \text{ (simplify on each side of equal side by collecting like terms)} \\
\frac{2}{3}x + \frac{8}{3} &= {\color{blue}\frac{11}{2}} - \frac{3}{4}x \\
&\text{(use Addition Principle to move all terms with } x \text{ to left side, all terms with constants to right side)} \\
\frac{2}{3}x {\color{blue}+ \frac{3}{4}x} + \cancel{\frac{8}{3}} {\color{red}- \cancel{\frac{8}{3}}} &= \frac{11}{2} {\color{red}- \frac{8}{3}} - \cancel{\frac{3}{4}x} {\color{blue}+ \cancel{\frac{3}{4}x}} \\
\frac{2}{3}x + \frac{3}{4}x &= \frac{11}{2} - \frac{8}{3} \text{ (now collect like terms)} \\
\frac{17}{12}x &= \frac{17}{6} \text{ (now use Multiplication Principle to isolate the } x\text{)} \\
\cancel{\frac{12}{17}} \times \cancel{\frac{17}{12}}x &= \frac{12}{17} \times \frac{17}{6} \text{ (simplify)} \\
x &= 2
\end{aligned}$$

Example 2.8.64 Solve $5(x-3) \leq 2(x-3)$.

The goal is still to first isolate a single term with the x in it on one side of the equation. If we multiply or divide by a negative number, we must switch the direction of the inequality.

$$\begin{aligned}
5(x-3) &\leq 2(x-3) \\
5x - 15 &\leq 2x - 6 \\
5x - \cancel{15} + \cancel{15} &\leq 2x - 6 + 15 \\
5x &\leq 2x + 9 \\
5x - 2x &\leq \cancel{2x} + 9 - \cancel{2x} \\
3x &\leq 9 \\
\frac{1}{3} \times 3x &\leq \frac{1}{3} \times 9 \\
x &\leq 3
\end{aligned}$$

Example 2.8.76 Solve $\frac{3x+5}{4}+\frac{7}{12}>-\frac{x}{6}$.

Let's start this one by clearing fractions. So we use the Multiplication Principle with the factor 12 (which is the LCD). Since 12 is positive, we don't change the direction of the inequality.

$$\begin{aligned}
\frac{3x+5}{4}+\frac{7}{12} &> -\frac{x}{6} \\
12\times\left(\frac{3x+5}{4}+\frac{7}{12}\right) &> 12\times\left(-\frac{x}{6}\right) \text{ (now distribute the factor of 12)} \\
3(3x+5)+7 &> -2x \text{ (distribute the 3)} \\
9x+15+7 &> -2x \text{ (simplify)} \\
9x+22 &> -2x \\
9x+22-9x &> -2x-9x \text{ (Use Additive Principle)} \\
22 &> -11x \\
&\text{(Use Multiplication Principle to isolate the } x\text{,} \\
&\text{ since we are multiplying by a negative number change direction of inequality)} \\
\frac{1}{-11}22 &< \frac{1}{-11}(-11x) \text{ (simplify)} \\
-2 &< x \text{ (simplify)}
\end{aligned}$$

Example Sharon sells sports cars, and can choose between \$100,000 or 8% of her sales as her salary. How much does she need to sell to make the 8% offer the better choice?

Let x be the amount that Sharon sells. We then want 8% of x to be greater than \$100,000.

$$\begin{aligned}
8\%x &> \$100,000 \\
0.08x &> \$100,000 \\
\frac{0.08x}{0.08} &> \frac{\$100,000}{0.08} \\
x &> \frac{\$100,000}{0.08} \\
x &> \$1,250,000
\end{aligned}$$

If Sharon can sell over \$1,250,000 worth of cars, she should take the 8% of her sales as her salary.

Example The number of pounds of fish caught by Jack was 813 pounds more than the amount of fish caught by Sally. The amount of fish caught by Ben was 623 pounds less than the amount caught by Sally. Write algebraic expressions for these relations.

Let the number of pounds of fish caught by Sally be x.
The number of pounds of fish caught by Jack is $x+813$.
The number of pounds of fish caught by Ben is $x-623$.

You can have different answers that are correct, but they are more complicated. Consider the following:

Let the number of pounds of fish caught by Jack be y.
Since Jack caught 813 pounds more than Sally, Sally caught 813 pounds less than Jack. The number of pounds of fish caught by Sally is $y-813$.
The number of pounds of fish caught by Ben is $y-813-623=y-1436$.

This is correct, but it required more translation of the English phrases than our first answer.

Example A hospital revealed that five officers of the hospital had an average salary of \$125,000 year. Three of the annual salaries are known to be \$50,000, \$60,000, and \$65,000. The annual salaries for the vice president and president were not revealed, but it is known that the president makes twice what the vice president makes. Find the salaries of the president and vice president.

Since there is a lot in the problem, I will use the Blueprint to organize my solution. When I was done filling in my Blueprint, I had nothing in the Key Points column, so I deleted it.

Gather Facts	Assign Variables	Basic Formula or Equation
• The average salary of the five executives is \$125,000. • We know three of the salaries, but don't know two. The ones we know are \$50,000, \$60,000, and \$65,000. The two we don't know are x and $2x$.	• Let x be the salary of the vice president. • Then $2x$ is the salary of the president.	• The average of five number is the sum divided by five. $\frac{\$50,000+\$60,000+\$65,000+x+2x}{5} = \$125,000$

Solve the equation for x:

$$\begin{aligned}\frac{\$50,000+\$60,000+\$65,000+x+2x}{5} &= \$125,000\\ \$175,000+3x &= \$625,000\\ 3x &= \$450,000\\ x &= \$150,000\end{aligned}$$

So the vice president earns \$150,000 and the president earns \$300,000.

Check: The average of the salaries should be \$125,000:

$$\frac{\$50,000+\$60,000+\$65,000+\$150,000+\$300,000}{5} = \frac{\$625,000}{5} = \$125,000\checkmark$$

Example In a triangle, the measure of the first angle is twice the measure of the second angle. The measure of the third angle is 20 degrees less than the second angle. What is the measure of each angle?

This does not require the Blueprint to solve.

Let the measure of the second angle be x degrees.
The measure of the first angle is $2x$ degrees.
The measure of the third angle is $x-20$ degrees.
The sum of the angles must equal 180 degrees: $x+2x+x-20=180$.

$$\Rightarrow 4x = 200 \Rightarrow x = 50$$

The first angle is 100 degrees, the second angle is 50 degrees, and the third angle is 30 degrees.

Check: $100+50+30=180\checkmark$

Example In her statistics course, Jill earned 80/100 and 75/100 on her two chapter tests. The chapter tests count 30% each towards her course grade, and the final exam is worth 40% of her course grade. What must Jill score on the final (out of 100) if she wishes to earn a final grade of at least 83/100?

Since there is a lot in the problem, I will use the Blueprint to organize my solution. When I was done filling in my Blueprint, I had nothing in the Key Points column, so I deleted it.

<u>Gather Facts</u>

- The chapter tests count 30% towards the course grade.
- Jill chapter test scores were 80 and 75 out of 100.
- The final exam counts 40% towards the course grade.

<u>Assign Variables</u>

- Let x be Jill's final exam score, out of 100.

<u>Basic Formula or Equation</u>

Jill's course grade (out of 100) will be equal to the following:

$30\%(80) + 30\%(75) + 40\%(x)$

To score above 83 in the course, she needs:

$30\%(80) + 30\%(75) + 40\%(x) > 83$

Solve the equation:

$$\begin{aligned} 30\%(80) + 30\%(75) + 40\%(x) &> 83 \\ \frac{30}{100}(80) + \frac{30}{100}(75) + \frac{40}{100}x &> 83 \\ (30)(80) + (30)(75) + 40x &> 8300 \\ 40x &> 3650 \\ x &> \frac{365}{4} \sim 91.25 \end{aligned}$$

Jill needs to score at least 92 out of 100 on the final (assuming she can't earn fractional points) to earn above 83/100 for the course grade.

Check: Assuming she earns 92% on the final, her grade will be

$$30\%(80) + 30\%(75) + 40\%(92) = 83.3 > 83\checkmark$$

Note: If all scores are not out of 100, you can work with fractions for every grade (including the final) to answer problems like this.

Example An ideal gas in state 1 has $P_1 = 2$ Pa, $V_1 = 20$ cm^3, and $T_1 = 12$ K. This gas is then adjusted so the temperature is $T_2 = 80$ K and the volume is $V_2 = 10$ cm^3. What is the pressure of the gas in state 2, using the Combined Gas Law?

$$\begin{aligned} \frac{P_1V_1}{T_1} &= \frac{P_2V_2}{T_2} \\ \frac{(2\text{Pa})(20\text{cm}^3)}{(12\text{K})} &= \frac{(P_2)(10\text{cm}^3)}{(80\text{K})} \\ P_2 &= \frac{(2\text{Pa})(20\text{cm}^3)(80\text{K})}{(12\text{K})(10\text{cm}^3)} \\ P_2 &= \frac{3200}{120}\text{Pa} = \frac{80}{3}\text{Pa} \sim 27\text{Pa} \end{aligned}$$

Example Solve the van der Waals equation (used to model fluid compression in chemistry) $\left(p + \frac{n^2a}{V^2}\right)(v - nb) = nRT$ for p.

$$\left(p + \frac{n^2a}{V^2}\right)(v - nb) = nRT$$

$$\left(p + \frac{n^2a}{V^2}\right)\cancel{(v - nb)} \times \frac{1}{\cancel{(v - nb)}} = nRT \times \frac{1}{(v - nb)} \text{ Division Principle}$$

$$p + \frac{n^2a}{V^2} = \frac{nRT}{(v - nb)} \text{ Simplify}$$

$$p + \cancel{\frac{n^2a}{V^2}} - \cancel{\frac{n^2a}{V^2}} = \frac{nRT}{(v - nb)} - \frac{n^2a}{V^2} \text{ Addition Principle (adding a negative quantity)}$$

$$p = \frac{nRT}{(v - nb)} - \frac{n^2a}{V^2} \text{ Simplify}$$

Note: Remember, other solutions are possible.