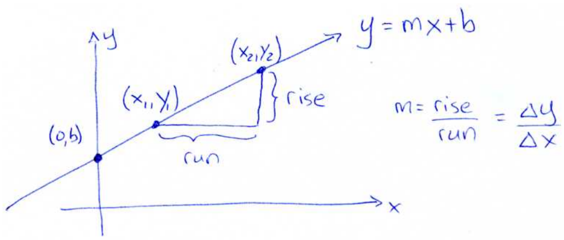
Sketching Straight Lines (Linear Relationships)



The *slope* of the line is $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\text{rise}}{\text{run}}$.

Horizontal lines have the form $y = b$ and have slope $m = 0$.
Vertical lines have the form $x = a$ and have infinite slope.

Parallel lines have the same slope.
Perpendicular lines have slopes whose product is -1.

The slope of a line represents a rate of change over an interval. This ides is captured in the notation $m = \dfrac{\Delta y}{\Delta x}$, where we read Δy as "the change in y" and Δx as "the change in x". You see this notation in physics and chemistry, where the Δ is used to represent an error in some measured quantity.

The equation of the line $y = mx + b$ represents an infinite set of ordered pairs. We can express this as $(x, y) = (x, mx + b)$ which explicitly shows this as a set of ordered pairs. If we pick a particular value of x, we can evaluate $(x, mx + b)$ to get an ordered pair.

Technique: To find specific ordered pairs, we can

- pick a value of x and use the equation to determine the corresponding value of y, or
- pick a value of y and use the equation to determine the corresponding value of x.

Example Find four ordered pairs that satisfy the equation $3x - 7y = -21$.

If $x = 0$, the the equation becomes $3(0) - 7y = -21 \Rightarrow y = 3$, so an ordered pair on the line is $(x, y) = (0, 3)$.

If $y = 0$, the the equation becomes $3x - 7(0) = -21 \Rightarrow x = -7$, so an ordered pair on the line is $(x, y) = (-7, 0)$.

If $y = -1$, the the equation becomes $3x - 7(-1) = -21 \Rightarrow x = -\frac{28}{3}$, so an ordered pair on the line is $(x, y) = \left(-\frac{28}{3}, -1\right)$.

If $x = 10$, the the equation becomes $3(10) - 7y = -21 \Rightarrow y = \frac{51}{7}$, so an ordered pair on the line is $(x, y) = \left(10, \frac{51}{7}\right)$.

When you are graphing an equation and inequality and not using graph paper (which we often do), here are some important things to do:

- Label your axes, x axis to the right (in the direction of increasing x) and y axis to the top (in the direction of increasing y).
- Include arrows on the ends of your line if the line continues forever.
- include the equation of the line somewhere on the graph beside the line.
- Explicitly label the points you used to create the line. I prefer not to use ticks on the axes, but you can use ticks if you want–but be neat!
- Any annotations you make on the graph (maybe a triangle that shows the slope between two points on the line) should be large and neatly labeled so it is easy to read.
- Make the entire graph large enough to easily read, and redraw it if necessary.

Technique: Plot two ordered pairs that satisfy the linear equation and draw a straight line through them. A third point that satisfies the linear equation can be used to check that the graph is correct.

Example Graph $3x + \frac{1}{3}y - 2 = -12$.

$3x + \frac{1}{3}y = -10$

If $x = 0$, $\frac{1}{3}y = -10$

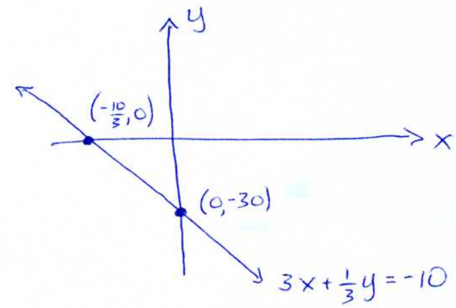
$y = -30$

$\Rightarrow (0, -30)$ is a point on line.

If $y = 0$, $3x = -10$

$x = -10/3$

$\Rightarrow (-\frac{10}{3}, 0)$ is a point on line.



One way to check this is to compute the slope from the sketch using the points you used and compare it to the slope from the equation (they should be equal!).

Three different ways of writing the equation of a line

- slope-intercept form, where m and y-intercept $(0, b)$ are given: $y = mx + b$.
- slope-point form, where m and point on the line (x_1, y_1) are given: $y - y_1 = m(x - x_1)$.
- point-point form, where two points on the line (x_1, y_1) and (x_2, y_2) are given: $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is sometimes written as $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$.

Choose one of the above to work with depending on what information you are given. The first two are the most used in mathematics, and you should strive to be comfortable working with both $y = mx + b$ and $y - y_1 = m(x - x_1)$.

Aside: Here is the derivation of the slope-point equation when you are given m and (x_1, y_1):

$$
\begin{aligned}
y &= mx + b && \text{(begin with slope-intercept equation)}\\
y_1 &= mx_1 + b && \text{(m is known, so substitute in $x = x_1$ and $y = y_1$)}\\
{\color{red}y_1 - mx_1} &= {\color{red}b} && \text{(solve for the unknown b)}\\
y &= mx + {\color{red}y_1 - mx_1} && \text{(substitute this value for b back in the original equation)}\\
y - y_1 &= mx - mx_1 && \text{(simplify)}\\
y - y_1 &= m(x - x_1) && \text{(factor–factoring is covered in Unit 7 in more detail)}
\end{aligned}
$$

The point-point equation follows from the slope-point equation by substituting $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Example Do the points $(2, 1)$, $(-3, -2)$ and $(7, 4)$ lie on the same line? If so, what is the equation of the line?

One way to check if they all lie on the same line is to check that the slope between all pairs of points is the same. Draw three points that aren't on the same line and see that this must the case.

Slope between $(2, 1)$ and $(-3, -2)$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{-2 - 1}{-3 - 2} = \dfrac{-3}{-5} = \dfrac{3}{5}$.

Slope between $(2, 1)$ and $(7, 4)$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{4 - 1}{7 - 2} = \dfrac{3}{5}$.

Slope between $(-3, -2)$ and $(7, 4)$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{4 - (-2)}{7 - (-3)} = \dfrac{6}{10} = \dfrac{3}{5}$.

Since the slopes between each pair of points is the same, the points all lie on the same line.

We can get the equation of the line using any of the formulas for equation of a line, so let's do that here so you can see how each gives the same final equation.

slope-intercept form:

$$\begin{aligned} y &= mx + b \\ y &= \frac{3}{5}x + b \quad \text{substitute in the slope which we worked out above} \\ (1) &= \frac{3}{5} \cdot (2) + b \quad \text{substitute in one of the points, here I've chosen } (x, y) = (2, 1) \\ 1 - \frac{6}{5} &= b \quad \text{solve for } b \\ -\frac{1}{5} &= b \\ y &= \frac{3}{5}x - \frac{1}{5} \quad \text{final equation for the line} \end{aligned}$$

slope-point form: Use $(x_1, y_1) = (2, 1)$ and $m = \frac{3}{5}$:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{3}{5}(x - 2) \quad \text{substitute in the slope and point} \\ y &= \frac{3}{5}x - \frac{6}{5} + 1 \quad \text{simplify to slope-intercept form to compare} \\ y &= \frac{3}{5}x - \frac{1}{5} \end{aligned}$$

point-point form: Use $(x_1, y_1) = (2, 1)$ and $(x_2, y_2) = (7, 4)$:

$$\begin{aligned} \frac{y - y_1}{y_2 - y_1} &= \frac{x - x_1}{x_2 - x_1} \\ \frac{y - 1}{4 - 1} &= \frac{x - 2}{7 - 2} \quad \text{substitute in the points} \\ \frac{y - 1}{3} &= \frac{x - 2}{5} \quad \text{simplify to slope-intercept form to compare} \\ 15 \times \left(\frac{y - 1}{3}\right) &= 15 \times \left(\frac{x - 2}{5}\right) \quad \text{(clear fractions)} \\ 5\,(y - 1) &= 3\,(x - 2) \\ 5y - 5 &= 3x - 6 \\ 5y &= 3x - 6 + 5 \\ 5y &= 3x - 1 \\ y &= \frac{3}{5}x - \frac{1}{5} \end{aligned}$$

Graphing linear inequalities

1. Replace the inequality symbol by an equality symbol. Graph the line.
   The line is solid if you had $\leq$ or $\geq$ and dashed if you had $<$ or $>$. The solid line means the line itself is part of the set of points that satisfies the inequality.

2. Test a point on one side of the line (the origin $(0, 0)$ is a good point to use as the test point if the line doesn't pass through it). If the inequality is true for the test point, shade the the side of the line that includes the test point. If the inequality is not true, shade the side of the inequality that does not contain the test point.

Example Sketch the region that satisfies the inequality $x - 5y > 3$.

1st sketch: $x - 5y = 3$

*line will be dashed since $>$

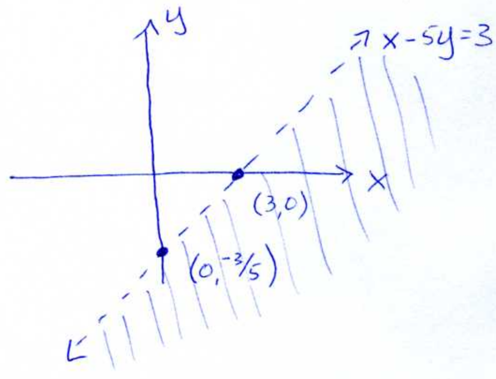
If $x = 0$, $-5y = 3$

$y = -3/5$

$\Rightarrow (0, -\frac{3}{5})$ is point on line

If $y = 0$, $x = 3$

$\Rightarrow (3,0)$ is a point on line



Test point $(0,0)$:

$x - 5y > 3$

$0 > 3$ False.

$\Rightarrow$ shade other side of line

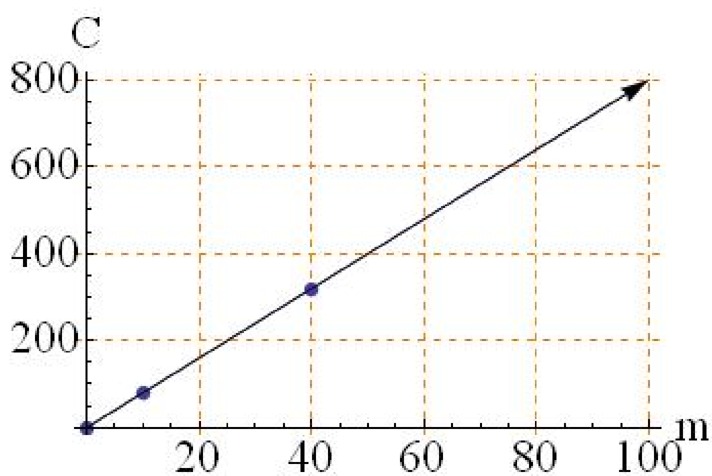
Example 3.2.29 The number of calories burned by an average person while cross-country skiing is given by the equation $C = 8m$, where m is the number of minutes. Graph the equation.

Note: The m in this equation is <u>not</u> the slope! The slope of the line is 8, and m is one of the variables.

To get the graph, we need at least two points on the line, and a third as a check.

If $m = 0$, $C = 8(0) = 0$ so a point on the line is $(0, 0)$.
If $m = 10$, $C = 8(10) = 80$ so a point on the line is $(10, 80)$.
If $m = 40$, $C = 8(40) = 320$ so a point on the line is $(40, 320)$.

Here is the sketch (I used a grid to mimic graph paper, you don't need to do that in your graphs):



Example 3.4.38 The amount of debt outstanding on home equity loans in the USA during the period from 1993 to 2008 can be approximated by the equation $y = mx + b$, where x is the number of years since 1993 and y is the debt measured in billions of dollars. Find the equation if two ordered pairs that satisfy it are $(1, 280)$ and $(6, 500)$.

We ultimately want a slope-intercept equation of a line, but we are given two points that lie on the line. Let's start with the point-point equation of a line, and use algebra to reduce it to a slope-intercept form. Choose $(x_1, y_1) = (1, 280)$ and $(x_2, y_2) = (6, 500)$.

$$\begin{aligned}
\frac{y-y_1}{y_2-y_1} &= \frac{x-x_1}{x_2-x_1}\\
\frac{y-280}{500-280} &= \frac{x-1}{6-1}\\
\frac{y-280}{220} &= \frac{x-1}{5}\\
220\times\left(\frac{y-280}{220}\right) &= 220\times\left(\frac{x-1}{5}\right) \text{ (clear fractions)}\\
y-280 &= 44\,(x-1)\\
y-280 &= 44x-44\\
y &= 44x-44+280\\
y &= 44x+236
\end{aligned}$$

Example A student organization sells t-shirts. When they charge \$15 per shirt, they sell 100 shirts. When they charge \$12 per shirt, they sell 175 shirts. Find a linear relation between the price of the shirts x and the number of shirts that are sold y.

Let the relation be $y = mx + b$. Our job is to figure out m and b.

Two points on the line are $(15, 100)$ and $(12, 175)$.

Slope $= m = \dfrac{\Delta y}{\Delta x} = \dfrac{100-175}{15-12} = \dfrac{-75}{3} = -25.$

Get b, the y-intercept:

$$\begin{aligned}
y &= mx+b\\
y &= -25x+b\\
100 &= -25(15)+b \text{ sub in a point on the line}\\
100 &= -375+b \text{ solve for } b\\
475 &= b
\end{aligned}$$

Relation is $y = -25x + 475$, or (number of shirts sold) $= -25$(price of shirt) $+ 475$.

Example Find the equation of the line perpendicular to the line $y = 3x$ that passes through the point $(-2, 1)$. Sketch the situation.

The new line should have slope $-\frac{1}{3}$ (perpendicular, so product of slopes should be -1).

Thus, the new line should look like $y = -\frac{1}{3}x + b$.

Now, use the point given to get the value of b.

$$
\begin{aligned}
y &= -\frac{1}{3}x + b \\
1 &= -\frac{1}{3}(-2) + b \\
1 &= \frac{2}{3} + b \\
1 - \frac{2}{3} &= b \\
\frac{1}{3} &= b \\
y &= -\frac{1}{3}x + \frac{1}{3} \text{ is the equation of the line we seek.}
\end{aligned}
$$

sketch $y=3x$

If $x=0$, $y=3(0)$
$y=0$

so $(0,0)$ is a point on the line.

choosing $y=0$ results in the same point, $(0,0)$, so use slope to get second point.

sketch $y=-\frac{1}{3}x+\frac{1}{3}$

If $x=0$, $y=-\frac{1}{3}(0)+\frac{1}{3}$
$y=\frac{1}{3}$

so $(0,\frac{1}{3})$ is a point on the line.

If $y=0$, $0=-\frac{1}{3}x+\frac{1}{3}$
$+\frac{1}{3}x=\frac{1}{3}$
$x=1$

so $(1,0)$ is a point on the line.

when sketching two lines that are supposed to be perpendicular, you should try to make the scale along x and y axes the same (or use graph paper).