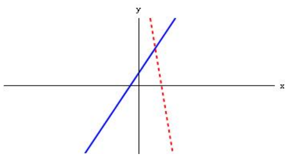
Solving Systems of linear equations in two unknown variables using algebra

Problems of this type look like: Solve the system of equations

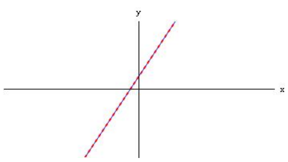
$$3x + 67y = 12$$
$$45x + 6y = -3$$

You will have one of three possibilities when solving two equations in two variables:

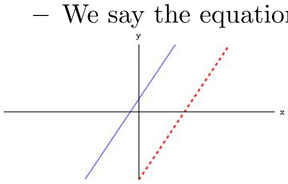
- You will find one ordered pair solution, (x, y).
  - This means the two equations represent two lines which intersect at the one point (x, y).



- You will find something like $12 = 12$ for example, which is true no matter what the variables are.
  - This means the two equations represent the same line.
  - This means that there is an infinite number of solutions.
  - We say the equations are dependent in this case.



- You end up with something like $2 = 5$, which is never true no matter what the variables are
  - This means the two equations represent two lines which are parallel.
  - This means that there is no solution.
  - We say the equations are inconsistent in this case.



There are two methods to do this, and I do not care which method you use.

Substitution Method

1. Choose one of the two equations and solve for one variable in terms of the other.
2. Substitute this expression from Step 1 into the *other equation*.
3. You now have one equation in one variable. Solve for the variable.
4. Substitute this value for the variable into one of the original equations to obtain the second variable.
5. Check the solution in both of the original equations.

Elimination Method

1. Arrange each equation in the form $ax + by = c$.
2. Multiply one or both equations by appropriate numbers so that that the coefficients of one of the variables are opposite (for example, -3 and $+3$).
3. Add the two equations from Step 2 so that one variable is eliminated.
4. Solve the resulting equation for the remaining variable.
5. Substitute this value for the variable into one of the original equations to obtain the second variable.
6. Check the solution in both of the original equations.

Solving Systems of Inequalities by sketching

The technique to do this just combines the techniques for graphing one inequality in two variables and finding the point of intersection of two lines.

Barry's Comments

The substitution method can be modified slightly to solve systems of 3 equations with 3 unknown variables (which we won't be studying here in this class), such as

$$\begin{aligned} 23x - 34y + 5z &= 89 \\ 7x - 6y + 22z &= 9 \\ -14x + 4y + z &= 0 \end{aligned}$$

Example Solve the system of equations

$$\begin{aligned} 3x + 67y &= 12 \\ 45x + 6y &= -3 \end{aligned}$$

Solution using substitution method:

Solve the first equation for x:

$$x = 4 - \frac{67}{3}y$$

Substitute into other equation:

$$45\left(4 - \frac{67}{3}y\right) + 6y = -3$$

Solve for y:

$$\begin{aligned} &45\left(4 - \frac{67}{3}y\right) + 6y = -3 \\ &180 - 1005y + 6y = -3 \\ &-999y = -183 \\ &y = \frac{-183}{-999} = \frac{61}{333} \end{aligned}$$

Substitute back into one of the earlier equations:

$$45x + 6\left(\frac{61}{333}\right) = -3$$

Solve for x:

$$\begin{aligned} &45x + \frac{122}{111} = -3 \\ &45x = -3 - \frac{122}{111} = -\frac{455}{111} \\ &x = -\frac{91}{999} \end{aligned}$$

The solution is $(x, y) = \left(-\frac{91}{999}, \frac{61}{333}\right)$.

Solution using elimination method:

Multiply the first equation by -15 and don't modify the second equation:

$$\begin{aligned} -45x - 1005y &= -180 \\ 45x + 6y &= -3 \end{aligned}$$

Add the two equations:

$$-999y = -183$$

Solve for y:

$$y = \frac{-183}{-999} = \frac{61}{333}$$

Substitute back into one of the earlier equations:

$$45x + 6\left(\frac{61}{333}\right) = -3$$

Solve for x:

$$\begin{aligned} &45x + \frac{122}{111} = -3 \\ &45x = -3 - \frac{122}{111} = -\frac{455}{111} \\ &x = -\frac{91}{999} \end{aligned}$$

The solution is $(x, y) = \left(-\frac{91}{999}, \frac{61}{333}\right)$.

Example Solve the system of equations

$$\begin{aligned} 2(y-3) &= x+3y \\ x+2 &= 3-y \end{aligned}$$

First, write both equations in the form $ax+by=c$.

$$\begin{aligned} \text{First Equation: } 2(y-3) &= x+3y \\ 2y-6 &= x+3y \\ -x-y &= 6 \\ \text{Second Equation: } x+2 &= 3-y \\ x+y &= 1 \end{aligned}$$

The system of equations is now:

$$\begin{aligned} -x-y &= 6 \\ x+y &= 1 \end{aligned}$$

Solve with the elimination method. Add the two equations.

$$0=7$$

This is never true. The system of equations has no solution.

Example Solve the system of equations

$$\begin{aligned} 21x-7y &= 14 \\ -21x+7y &= -14 \end{aligned}$$

Solve the first equation for x: $x=\dfrac{2}{3}+\dfrac{1}{3}y$.

Substitute into the second equation, and then solve for y:

$$\begin{aligned} -21x+7y &= -14 \\ -21\left(\frac{2}{3}+\frac{1}{3}y\right)+7y &= -14 \\ -14-7y+7y &= -14 \\ -14 &= -14 \end{aligned}$$

This is true no matter what x is. There are an infinite number of ordered pairs that satisfy the system of equations.

If we want to express the infinite number of ordered pairs, here is one way to do it: The two equations in the system are actually *equivalent equations*, so we can pick either one of them as the representation of the infinite number of ordered pairs. Let's pick the first, $21x-7y=14$. This equation ($21x-7y=14$) represents all the ordered pairs that satisfy the system of equations.

If we want to make it a bit clearer that this represents an infinite set of ordered pairs, we can solve for y:

$$\begin{aligned} 21x-7y &= 14 \\ -7y &= 14-21x \\ y &= -2+3x \end{aligned}$$

and we can express the set of ordered pairs as $(x,y)=(x,-2+3x)$.

Aside: What we have done in this last step is create a *parametric relation*, since we could also write the ordered pairs as $(t, -2+3t)$ and we get different ordered pairs by picking different values of the parameter t. You will see parametric relations in precalculus.

Example The Tupper Farm has 450 acres of land allotted for raising corn and wheat. The cost to cultivate corn in \$42 per acre. The cost to cultivate wheat is \$35 per acre. The Tuppers have \$16,520 available to cultivate these crops. How many acres of each crop should the Tuppers plant?

Begin by defining your variables.

x is the amount of corn they plant.

y is the amount of wheat they plant.

Use the information given to get two equations:

450 acres of land allotted for raising corn and wheat	means	$x + y = 450.$
The cost to cultivate corn in \$42 per acre. The cost to cultivate wheat is \$35 per acre. The Tuppers have \$16,520 available to cultivate these crops.	means	$\$42x + \$35y = \$16,520.$

The two equations we need to solve are

$$\begin{aligned} x + y &= 450 \\ 42x + 35y &= 16,520 \end{aligned}$$

Solve with the elimination method. multiply the first equation by -35:

$$\begin{aligned} -35x - 35y &= -15750 \\ 42x + 35y &= 16,520 \end{aligned}$$

Add the equations

$$7x = 770 \Rightarrow x = 110.$$

Substitute back into an earlier equation and solve for y:

$$x + y = 450 \Rightarrow 110 + y = 450 \Rightarrow y = 450 - 110 = 340.$$

The Tuppers can plant 110 acres of corn and 340 acres of wheat.

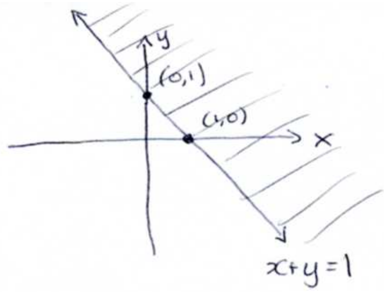
Example Sketch the region that satisfies the following inequalities:

$x + y \geq 1$
$x - 3y \geq 2$
$x > 2$

$x + y \geq 1$

sketch $x+y=1$
when $x=0, y=1 \Rightarrow (0,1)$
when $y=0, x=1 \Rightarrow (1,0)$

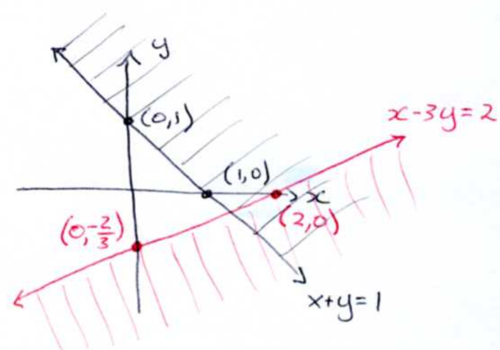
Test point $(0,0)$:
$0+0 \geq 1$
$0 \geq 1$ False.



$x - 3y \geq 2$

sketch $x-3y = 2$
when $x=0, -3y=2 \Rightarrow y=-\frac{2}{3} \Rightarrow (0,-\frac{2}{3})$
when $y=0, x=2 \Rightarrow 0\ (2,0)$

Test point $(0,0)$:
$0-0 \geq 2$
$0 \geq 2$ False



$x > 2$

sketch $x=2$. Vertical line at $x=2$.

$x>2$ will be to the right of this line.