Exponents

- Rules of Exponents:
  - $x^0 = 1$ if $x \neq 0$ (0^0 is indeterminant and is dealt with in calculus).
  - Product Rule: $x^a \cdot x^b = x^{a+b}$.
  - Quotient Rule: $\dfrac{x^a}{x^b} = x^{a-b}$.
  - Power Rule: $(x^a)^b = x^{ab}$.
  - Product Raised to Power Rule: $(xy)^a = x^a y^a$.
  - Quotient Raised to a Power Rule: $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$ if $y \neq 0$.
  - Negative Exponent: $x^{-n} = \dfrac{1}{x^n}$, if $x \neq 0$.

The rules of exponents are *tremendously important*, and it is critical that you memorize them. They will be useful in a variety of ways for the rest of the course and beyond.

Notice I did not break the quotient rule up into a bunch of cases–that isn't necessary if you are comfortable working with negative exponents (which you will be!).

Example Simplify by combining exponents in $\left(\dfrac{3x^{-3}y^{-2}}{x^{-2}y^{-4}}\right)^2$. Make all exponents positive in your final answer.

$$\left(\frac{3x^{-3}y^{-2}}{x^{-2}y^{-4}}\right)^2 = \frac{3^2(x^{-3})^2(y^{-2})^2}{(x^{-2})^2(y^{-4})^2} = \frac{9x^{-6}y^{-4}}{x^{-4}y^{-8}} = 9x^{-6+4}y^{-4+8} = 9x^{-2}y^4 = \frac{9y^4}{x^2}$$

Polynomials

- A polynomial is the sum of a finite number of terms of the form ax^n where a is any real number and n is a whole number.
- A multivariable polynomial has more than one polynomial, for example $56x^3y^4 + xy^2 - x^2$.
- The degree of a term is the sum of the exponents of all the variables in the term.
- The degree of a polynomial is the largest degree of all the terms in the polynomial.

Add or subtract two polynomials by collecting like terms.

Multiply polynomials by using the distributive property.

$$\begin{aligned}(x + 2x^2 + 4x^3)(1 + y) &= x(1 + y) + 2x^2(1 + y) + 4x^3(1 + y) \text{ (distribute the factor (1+y) into the first polynomial)}\\ &= x + xy + 2x^2 + 2x^2y + 4x^3 + 4x^3y \text{ (now distribute the factor into (1+y))}\end{aligned}$$

you could then collect like terms to simplify–in this case, there are no like terms so we are done.

Special cases of multiplication (these occur frequently, so they are useful to know, but you could always work these out using the distributive property):

- $(a + b)(a - b) = a^2 - b^2$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$

The text talks about a vertical multiplication of polynomials on page 353. Use that if you like, it is mathematically correct. But I far prefer using the distributive property to multiply polynomials, and I think you will too with a bit of practice. Add as many intermediate steps as you need to get the correct final answer!

Example 5.1.40 Multiply $(3ab)(5a^2c)(-2b^2c^3)$.

$$\begin{aligned}(3ab)(5a^2c)(-2b^2c^3) &= -(3ab)(5a^2c)(2b^2c^3) && \text{(deal with the overall sign first)}\\ &= -(3\cdot 5\cdot 2)(aa^2)(bb^2)(cc^3) && \text{(collect together constants and factors with the same base)}\\ &= -(30)(a^3)(b^3)(c^4) && \text{(simplify using rules of exponents–remember } a = a^1\text{)}\\ &= -30a^3b^3c^4 && \text{(remove extraneous parentheses)}\end{aligned}$$

Example Multiply $(9a - 14b)^2$.

$$\begin{aligned}(9a-14b)^2 &= (9a-14b)(9a-14b) \text{ (I am going to use the distributive property, so expand the exponent)}\\ &= 9a(9a-14b) - 14b(9a-14b) \text{ (here I have distributed the } 9a-14b \text{ factor)}\\ &= 9a(9a) - 9a(14b) + (-14b)9a - (-14b)14b \text{ (here I have distributed the } 9a \text{ factor and the } -14b \text{ factors)}\\ &= 81a^2 - 126ab - 126ab + 196b^2 \text{ (simplify)}\\ &= 81a^2 - 256ab + 196b^2 \text{ (done)}\end{aligned}$$

Example Multiply $(x^2 - 5x)(x^5 + 7x)(x^3 + x + 3)$.

This is a test of your ability to keep tracks of all the pieces during a long solution.

Multiply the first two factors together:

$$\begin{aligned}(x^2-5x)(x^5+7x) &= (x^2-5x)(x^5) + (x^2-5x)(7x) && \text{(distribute the } x^2-5x \text{ factor)}\\ &= x^2(x^5) - 5x(x^5) + x^2(7x) - 5x(7x) && \text{(distribute the factors into the } x^2-5x\text{)}\\ &= x^{2+5} - 5x^{1+5} + 7x^{2+1} - 35x^{1+1} && \text{(simplify using exponent rule } x^a\cdot x^b = x^{a+b}\text{)}\\ &= x^7 - 5x^6 + 7x^3 - 35x^2 && \text{(simplify)}\end{aligned}$$

Now for the final polynomial product:

$$\begin{aligned}(x^2-5x)(x^5+7x)(x^3+x+3) &= (x^7 - 5x^6 + 7x^3 - 35x^2)(x^3+x+3)\\ &\text{(distribute the } x^3+x+3 \text{ factor)}\\ &= x^7(x^3+x+3) - 5x^6(x^3+x+3) + 7x^3(x^3+x+3) - 35x^2(x^3+x+3)\\ &\text{(distribute the factors into the } x^3+x+3\text{)}\\ &= (x^7)x^3 + (x^7)x + (x^7)3 + (-5x^6)x^3 + (-5x^6)x + (-5x^6)3 + (7x^3)x^3\\ &+ (7x^3)x + (7x^3)3 + (-35x^2)x^3 + (-35x^2)x + (-35x^2)3\\ &\text{(simplify using exponent rule } x^a\cdot x^b = x^{a+b}\text{)}\\ &= x^{7+3} + x^{7+1} + 3x^7 - 5x^{6+3} - 5x^{6+1} - 15x^6 + 7x^{3+3}\\ &+ 7x^{3+1} + 21x^3 - 35x^{2+3} - 35x^{2+1} - 105x^2\\ &= x^{10} + x^8 + 3x^7 - 5x^9 - 5x^7 - 15x^6 + 7x^6 + 7x^4 + 21x^3 - 35x^5 - 35x^3 - 105x^2\\ &\text{(collect like terms)}\\ &= x^{10} - 5x^9 + x^8 - 2x^7 - 8x^6 - 35x^5 + 7x^4 - 14x^3 - 105x^2\end{aligned}$$

- The order you distribute in when multiplying polynomials does not matter.

$$\begin{aligned}(x^2-4x+5)(x-2) &= (x^2-4x+5)x + (x^2-4x+5)(-2) \text{ (distribute the } x^2-4x+5 \text{ term)}\\ &= x^3 - 4x^2 + 5x + x^2(-2) - 4x(-2) + 5(-2) \text{ (now distribute the factors into the } x^2-4x+5\text{)}\\ &= x^3 - 4x^2 + 5x - 2x^2 + 8x - 10 \text{ (simplify)}\\ &= x^3 {\color{red}- 4x^2} {\color{blue}+ 5x} {\color{red}- 2x^2} {\color{blue}+ 8x} - 10 \text{ (collect like terms–I've used colour here)}\\ &= x^3 - 6x^2 + 13x - 10 \text{ (done)}\end{aligned}$$

Here's an alternate simplification:

$$
\begin{aligned}
(x^2 - 4x + 5)(x - 2) &= x^2(x - 2) - 4x(x - 2) + 5(x - 2) \text{ (distribute the } x - 2 \text{ term)} \\
&= x^3 - 2x^2 - 4x^2 + 8x + 5x - 10 \text{ (now distribute the factors into the } x - 2) \\
&= x^3 {\color{red} - 2x^2 - 4x^2} {\color{blue} + 8x + 5x} - 10 \text{ (collect like terms–I've used colour here)} \\
&= x^3 - 6x^2 + 13x - 10 \text{ (done)}
\end{aligned}
$$

• FOIL obscures the distributive property you are using, and only works on binomials, so I don't use it. The distributive property is what I use. You are, as always, free to use it since FOIL is mathematically correct.

Dividing polynomials is best done by the long division form discussed in Section 5.6. Go slow and be careful at each step–it's easy to make mistakes with this process! Practice will make the steps clear.

• Some of you may have learned the process called *synthetic division* to divide two polynomials. You are welcome to use this if you like. I have never used synthetic division in my life, and feel that it obscures what is actually happening when you divide two polynomials. Synthetic division also does not allow you to divide by quantities like $3x^2 + 1$, which long division does without difficulty. Finally, long division of polynomials is much easier for a reader to follow, and therefore much better for us to do!

Example Divide $\dfrac{6y^3 - 3y^2 + 4}{2y + 1}$.

Since we are missing a term, we need to remember to write the numerator as $6y^3 - 3y^2 + 0y + 4$.

$$
\begin{array}{r|l}
 & \; 3y^2 - 3y + \frac{3}{2} \\
2y+1 & \overline{6y^3 - 3y^2 + 0y + 4} \\
 & \underline{6y^3 + 3y^2 } \quad \text{(subtract)} \\
 & -6y^2 + 0y + 4 \\
 & \underline{-6y^2 - 3y } \quad \text{(subtract)} \\
 & 3y + 4 \\
 & \underline{3y + \tfrac{3}{2}} \quad \text{(subtract)} \\
 & \tfrac{5}{2} \leftarrow \text{remainder}
\end{array}
$$

Our long division tells us that

$$\frac{6y^3 - 3y^2 + 4}{2y + 1} = 3y^2 - 3y + \frac{3}{2} + \frac{5/2}{2y + 1}.$$

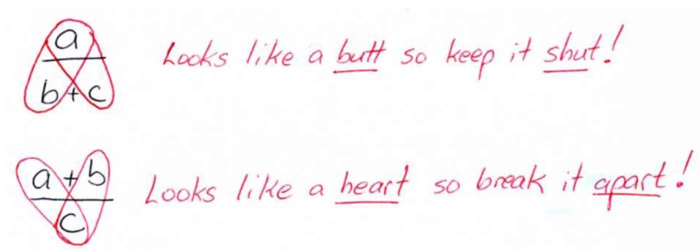
We can check if this is correct by finding a common denominator on the right hand side, which will involve polynomial multiplication:

$$
\begin{aligned}
3y^2 - 3y + \frac{3}{2} + \frac{5/2}{2y+1} &= \left(3y^2 - 3y + \frac{3}{2}\right)\frac{2y+1}{2y+1} + \frac{\frac{5}{2}}{2y+1} \\
&= \frac{\left(3y^2 - 3y + \frac{3}{2}\right)(2y+1) + \frac{5}{2}}{2y+1} \\
&= \frac{3y^2(2y+1) - 3y(2y+1) + \frac{3}{2}(2y+1) + \frac{5}{2}}{2y+1} \\
&= \frac{3y^2(2y) + 3y^2(1) + (-3y)2y + (-3y)1 + \frac{3}{2}(2y) + \frac{3}{2} + \frac{5}{2}}{2y+1} \\
&= \frac{6y^3 + 3y^2 - 6y^2 + 4}{2y+1} \\
&= \frac{6y^3 - 3y^2 + 4}{2y+1}
\end{aligned}
$$

which verifies the long division result. Obviously, checking your long division in this manner gives you good practice with polynomial multiplication as well!

Fun Fact

Someone once showed me the following:



which should help us remember that $\dfrac{a}{b+c} \neq \dfrac{a}{b} + \dfrac{a}{c}$, but we can write $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$.

• Appendix A.2 has some further discussion on polynomial operations.

• It is important to be good at multiplication of polynomials, since in Unit 7 we will look at factoring of polynomials, which is in essence the opposite of multiplication of polynomials (and yes, it is related to polynomial division).