Special Cases of Factoring

Difference of Two Squares

$$a^2 - b^2 = (a - b)(a + b).$$

Perfect Square (two cases)

$$a^2 + 2ab + b^2 = (a + b)^2,$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

Sum and Difference of Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

These should be used as formulas you have memorized, not something you "work out". You can "work out" the difference of square and perfect square cases using the grouping method of factoring if you have to, but the sum and difference of cubes must be used as memorized formulas.

Example Factor $27x^3 + \frac{729}{8}y^3$.

Notice that this looks like it might be a sum of cubes. Try to write in terms of quantities cubed:

$$27x^3 + \frac{729}{8}y^3 = (3x)^3 + \left(\frac{9}{2}y\right)^3$$

Now, use the appropriate formula you have memorized: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. Identify: $a = 3x$, $b = \frac{9}{2}y$.

$$\begin{aligned}
27x^3 + \frac{729}{8}y^3 &= (3x)^3 + \left(\frac{9}{2}y\right)^3 \\
&= (a)^3 + (b)^3 \quad \text{identify is as sum of cubes} \\
&= (a + b)(a^2 - ab + b^2) \quad \text{use sum of cubes formula} \\
&= \left(3x + \frac{9}{2}y\right)\left((3x)^2 - (3x)(\frac{9}{2}y) + (\frac{9}{2}y)^2\right) \quad \text{sub back in } a = 3x \text{ and } b = \frac{9}{2}y \\
&= \left(3x + \frac{9}{2}y\right)\left(9x^2 - \frac{27}{2}xy + \frac{81}{4}y^2\right) \quad \text{simplify}
\end{aligned}$$

I would accept the above since it is factored, but remember that you can sometimes write a factored form in different ways. The following are all equivalent:

$$\begin{aligned}
27x^3 + \frac{729}{8}y^3 &= \left(3x + \frac{9}{2}y\right)\left(9x^2 - \frac{27}{2}xy + \frac{81}{4}y^2\right) \\
&= 3\left(x + \frac{3}{2}y\right)\left(9x^2 - \frac{27}{2}xy + \frac{81}{4}y^2\right) \\
&= 3\left(x + \frac{3}{2}y\right)9\left(x^2 - \frac{3}{2}xy + \frac{9}{4}y^2\right) \\
&= 27\left(x + \frac{3}{2}y\right)\left(x^2 - \frac{3}{2}xy + \frac{9}{4}y^2\right) \\
&= 27\left(\frac{2x + 3y}{2}\right)\left(\frac{4x^2 - 6xy + 9y^2}{4}\right) \\
&= \frac{27}{8}(2x + 3y)(4x^2 - 6xy + 9y^2)
\end{aligned}$$

Example Factor $9x^2 - 42xy + 49y^2$.

Notice that the first term and last term look like squares:
$9x^2 = (3x)^2 = a^2$ (so $a = 3x$) and $49y^2 = (7y)^2 = b^2$ (so $b = 7y$).
Check if the middle term is $2ab = 2(3x)(7y) = 42xy$. Yes! So this is a perfect square (difference).

$$\begin{aligned} 9x^2 - 42xy + 49y^2 &= a^2 - 2ab + b^2 \quad \text{identify as a perfect square (difference)} \\ &= (a-b)^2 \quad \text{perfect square (difference) formula} \\ &= (3x - 7y)^2 \quad \text{sub back in } a = 3x \text{ and } b = 7y. \end{aligned}$$

Example Factor $\dfrac{1}{256} + \dfrac{11}{40}x + \dfrac{121}{25}x^2$.

Notice that the first term and last term look like squares:
$\dfrac{1}{256} = \left(\dfrac{1}{16}\right)^2 = a^2$ (so $a = \frac{1}{16}$) and $\dfrac{121}{25}x^2 = \left(\dfrac{11}{5}x\right)^2 = b^2$ (so $b = \frac{11}{5}x$).
Check if the middle term is $2ab = 2\left(\dfrac{1}{16}\right)\left(\dfrac{11}{5}x\right) = \dfrac{11}{40}x$. Yes! So this is a perfect square (sum).

$$\begin{aligned} \frac{1}{256} + \frac{11}{40}x + \frac{121}{25}x^2 &= a^2 + 2ab + b^2 \quad \text{identify as a perfect square (sum)} \\ &= (a+b)^2 \quad \text{perfect square (sum) formula} \\ &= \left(\frac{1}{16} + \frac{11}{5}x\right)^2 \quad \text{sub back in } a = \frac{1}{16} \text{ and } b = \frac{11}{5}x. \end{aligned}$$

Example Factor $16x^4 - 1$.

$$\begin{aligned} 16x^4 - 1 &= \left(4x^2\right)^2 - (1)^2 \quad \text{rewrite to see if it is a difference of squares} \\ &= (a)^2 - (b)^2 \quad \text{Identify as difference of squares, } a = 4x^2,\ b = 1 \\ &= (a+b)(a-b) \quad \text{write down memorized formula} \\ &= (4x^2+1)(4x^2-1) \quad \text{substitute back values for } a = 4x^2 \text{ and } b = 1 \\ &= (4x^2+1)((2x)^2 - (1)^2) \quad \text{first polynomial is prime; second is a difference of squares, } a = 2x,\ b = 1 \\ &= (4x^2+1)(a+b)(a-b) \quad \text{write down memorized formula} \\ &= (4x^2+1)(2x+1)(2x-1) \quad \text{substitute back values for } a = 2x \text{ and } b = 1 \end{aligned}$$

Example Factor $\dfrac{1}{16}r^2 - \dfrac{13}{2}rt + 169t^2$.

Notice that the first term and last term look like squares:
$\dfrac{1}{16}r^2 = \left(\dfrac{1}{4}r\right)^2 = a^2$ (so $a = \frac{1}{4}r$) and $169t^2 = (13t)^2 = b^2$ (so $b = 13t$).
Check if the middle term is $2ab = 2\left(\dfrac{1}{4}r\right)(13t) = \dfrac{13}{2}rt$. Yes! So this is a perfect square (difference).

$$\begin{aligned} \frac{1}{16}r^2 - \frac{13}{2}rt + 169t^2 &= a^2 - 2ab + b^2 \quad \text{identify as a perfect square (difference)} \\ &= (a-b)^2 \quad \text{perfect square (difference) formula} \\ &= \left(\frac{1}{4}r + 13t\right)^2 \quad \text{sub back in } a = \frac{1}{4}r \text{ and } b = 13t. \end{aligned}$$

A Prime Polynomial is a polynomial that cannot be factored using the techniques we developed here. For now, we are concerned with integers, so something like $x^2 - 3$ would be considered a prime polynomial even though we can write it as $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$.

In practice you aren't told which factoring technique to use, so being able to recognize the different cases is important.

Using Factoring to Solve Equations

Zero factor property: If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Solve quadratic equations by factoring:

$2x^2 + x - 10 = 0$

Look for two numbers whose product is -20 and sum is $+1$: $+5, -4$

$2x^2 + 5x - 4x - 10 = 0$

$x(2x + 5) - 2(2x - 5) = 0$

$(x - 2)(2x + 5) = 0$

$x - 2 = 0$ or $2x + 5 = 0$

$x = 2$ or $x = -5/2$

A very common error in solving quadratic equations $ax^2 + bx + c = 0$ is to try to "isolate" the x (since this is what you did for linear equations). This is entirely the wrong way to proceed, and will not lead you to the solution.

What this error looks like is the following:

$$\begin{aligned} 3x^2 + 5x + 2 &= 0 \text{ (equation we want to solve for } x) \\ 3x^2 + 5x &= -2 \\ x(3x + 5) &= -2 \\ 3x + 5 &= -\frac{2}{x} \\ 3x &= -\frac{2}{x} - 5 \\ x &= -\frac{2}{3x} - \frac{5}{3} \\ &\vdots \end{aligned}$$

and so on. These steps are all mathematically correct, but they will not lead to the solution since your goal when solving a quadratic equation is not to isolate the x. Compare the above to a correct solution that uses factoring:

$$\begin{aligned} 3x^2 + 5x + 2 &= 0 \text{ (equation we want to solve for } x) \\ & \quad \text{(factor by grouping method: need two numbers whose product is 6 and sum is 5: 2 and 3)} \\ 3x^2 + 2x + 3x + 2 &= 0 \text{ (rewrite middle term using 2 and 3)} \\ x(3x + 2) + (3x + 2) &= 0 \text{ (common factor in terms)} \\ (x + 1)(3x + 2) &= 0 \text{ (factor by grouping)} \\ (x + 1) = 0 \text{ or } (3x + 2) &= 0 \text{ (zero factor property)} \\ x = -1 \text{ or } x &= -\frac{2}{3} \text{ (simplify)} \end{aligned}$$

Application: Falling Objects When an object is thrown straight upwards, it's height S in meters is given by the quadratic equation

$$S = -\frac{g}{2}t^2 + vt + h,$$

$$\begin{aligned} \text{where } g &= \text{ acceleration due to gravity, which is } 9.8\text{m/s}^2 \\ v &= \text{ initial upward velocity of the object in m/s} \\ h &= \text{ height above ground from which the object is thrown in meters} \\ t &= \text{ time in seconds} \end{aligned}$$

This mathematical model of a physical system assumes there is no wind resistance (among other things). We often approximate the model as

$$S = -5t^2 + vt + h.$$

Example Jon is in a baseball park outfield stands (right by the field), and he throws a ball straight up in the air with all his might. It leaves his hand with a velocity of 13 m/s, and at a height of 6 m above the field. How long will it take the ball to land on the field?

Solution: The ball hits the field when $S = 0$, and we also have $v = 13$ m/s and $h = 6$ m, so we need to solve

$$\begin{aligned} 0 = -5t^2 + 13t + 6 &\text{ (grouping method: need two numbers whose product is } -30 \text{ and sum is 13: 15 and } -2) \\ 0 = -5t^2 + 15t - 2t + 6 &\text{ (rewrite middle term using 15 and } -2) \\ 0 = 5t(-t+3) + 2(-t+3) &\text{ (common factors in terms)} \\ 0 = (5t+2)(-t+3) &\text{ (factor by grouping)} \\ (5t+2) = 0 \text{ or } (-t+3) = 0 &\text{ (zero factor property)} \\ t = -\frac{2}{5} \text{ or } t = 3 &\text{ (simplify)} \end{aligned}$$

Exclude the negative answer as unphysical, and we see the ball hits the field after 3 seconds.

Example You are standing on a cliff overlooking the ocean. You are 180 meters above the ocean. You drop a pebble into the ocean (dropping implies the initial velocity is 0 m/s). How long will it take for the pebble to hit the water?

Solution: The pebble hits the water when $S = 0$, and we also have $v = 0$ m/s and $h = 180$ m, so we need to solve

$$\begin{aligned} 0 &= -5t^2 + 180 \\ 0 &= -5(t^2 - 36) \text{ (common factors in terms)} \\ \frac{0}{-5} &= \frac{-5(t^2-36)}{-5} \text{ (divide each side by } -5) \\ 0 &= t^2 - 36 \quad \text{(difference of squares)} \\ 0 &= (t-6)(t+6) \\ (t-6) = 0 &\text{ or } (t+6) = 0 \quad \text{(zero factor property)} \\ t = 6 &\text{ or } t = -6 \quad \text{(simplify)} \end{aligned}$$

Exclude the $t = -6$ as unphysical, and we see the pebble hits the water after 6 seconds.

Example Factor $3x^3a^3 - 11x^4a^2 - 20x^5a$.

$$\text{(common factors in terms) } 3x^3a^3 - 11x^4a^2 - 20x^5a = x^3a(3a^2 - 11xa - 20x^2)$$

To factor $3a^2 - 11xa - 20x^2$ we can treat this one of two ways:

1. As a quadratic in x: $-20x^2 - 11ax + 3a^2$
2. As a quadratic in a: $3a^2 - 11xa - 20x^2$

Let's do the first, and see what happens. Let a tag along as if it was a number, and use the grouping method.

The grouping number is $(-20)(3a^2) = -60a^2$.

Find two numbers whose product is $-60a^2$ and whose sum is $-11a$: $-15a$ and $4a$.

We will use these numbers to rewrite the middle term.

$$\begin{aligned}
-20x^2 - 11xa + 3a^2 &= -20x^2 - 15ax + 4ax + 3a^2 \\
&= -5x(4x + 3a) + a(4x + 3a) && \text{(common factors in terms)} \\
&= (-5x + a)(4x + 3a) && \text{(factor by grouping)}
\end{aligned}$$

Putting this back, we have determined that

$$3x^3a^3 - 11x^4a^2 - 20x^5a = x^3a(-5x + a)(4x + 3a).$$

Example The area of a triangle is 3 ft^2. The height is 10 ft longer than 4 times the base. Determine the dimensions of the triangle.

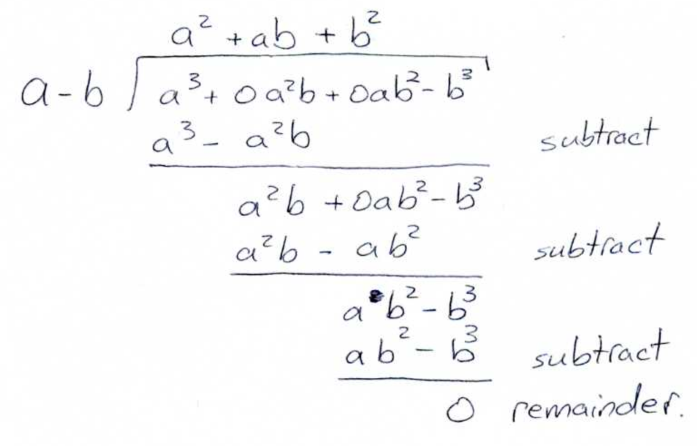
Let x be the length of the base. Then the height is equal to $4x + 10$.

$$\begin{aligned}
\text{Area} &= \frac{1}{2}(\text{height})(\text{base}) \\
3 &= \frac{1}{2}(4x + 10)(x) \\
6 &= (4x + 10)(x) \\
4x^2 + 10x - 6 &= 0 \text{ Two numbers: product } -24\text{, sum is 10: } 12, -2 \\
4x^2 + 12x - 2x - 6 &= 0 \\
4x(x + 3) - 2(x + 3) &= 0 \\
(4x - 2)(x + 3) &= 0 \\
4x - 2 = 0 &\text{ or } x + 3 = 0 \\
x = 1/2 &\text{ or } x = -3
\end{aligned}$$

Exclude the $x = -3$ as unphysical, and the dimensions of the triangle are base $= 1/2$ ft, and height $= 4(1/2) + 10 = 12$ ft.

Example By Dividing $a - b$ into $a^3 - b^3$, verify the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

Before we can divide, we have to include missing terms: $a^3 - b^3 = a^3 + 0a^2b + 0ab^2 - b^3$.



This shows that

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example Factor $x^4 - y^4$.

$$\begin{aligned}
x^4 - y^4 &= \left(x^2\right)^2 - (y^2)^2 \text{ rewrite to see if it is a difference of squares} \\
&= (a)^2 - (b)^2 \text{ Identify as difference of squares, } a = x^2,\ b = y^2 \\
&= (a + b)(a - b) \text{ write down memorized formula} \\
&= (x^2 + y^2)(x^2 - y^2) \text{ substitute back values for } a = x^2 \text{ and } b = y^2 \\
&= (x^2 + y^2)(x^2 - y^2) \text{ first polynomial is prime; second is a difference of squares, } a = x,\ b = y \\
&= (x^2 + y^2)(a + b)(a - b) \text{ write down memorized formula} \\
&= (x^2 + y^2)(x + y)(x - y) \text{ substitute back values for } a = x \text{ and } b = y
\end{aligned}$$