Rational Expressions

A rational expression is a polynomial divided by another polynomial.

Factoring continues to be an important technique when dealing with rational expressions. You are also going to be using the same techniques from Unit 1, but now on polynomials instead of integers. The text does a good job of showing the similarities.

The denominator in a rational expression cannot equal zero. We exclude any values that make a denominator zero.

To simplify rational expressions you must separately factor the numerator and denominator, and then cancel using the Basic Rule of Fractions:

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{c} \text{ if } b \neq 0 \text{ and } c \neq 0, \text{ where } a, b \text{ and } c \text{ are polynomials.}$$

$$\frac{x^2 - 1}{x^2 - 3x - 4} = \frac{\cancel{(x+1)}(x-1)}{(x-4)\cancel{(x+1)}} = \frac{x-1}{x-4} \text{ as long as } x + 1 \neq 0.$$

The basic idea of factoring and canceling terms from the numerator and denominator is the same as we used for integer fractions in Unit 1. For example, compare the following:

$$\begin{aligned}
\frac{x^2 - 5x + 6}{2x^9 - 18} &= \frac{(x-2)(x-3)}{2(x-3)(x+3)} \text{ factor the polynomials} \\
&= \frac{(x-2)\cancel{(x-3)}}{2\cancel{(x-3)}(x+3)} \text{ cancel between the numerator and denominator} \\
&= \frac{(x-2)}{2(x+3)}, \ x - 3 \neq 0
\end{aligned}$$

$$\begin{aligned}
\frac{2310}{440} &= \frac{(2)(3)(5)(7)(11)}{(2)(2)(2)(5)(11)} \text{ factor the integers} \\
&= \frac{\cancel{(2)}(3)\cancel{(5)}(7)\cancel{(11)}}{\cancel{(2)}(2)(2)\cancel{(5)}\cancel{(11)}} \text{ cancel between the numerator and denominator} \\
&= \frac{(3)(7)}{(2)(2)} \\
&= \frac{21}{4}
\end{aligned}$$

To multiply rational expressions, multiply the numerators and denominators, being careful to use parentheses where needed:

$$\begin{aligned}
\frac{x^2 - 3x + 2}{x^2 - 81} \cdot \frac{x - 9}{x^2 + 5x + 6} &= \frac{(x^2 - 3x + 2)(x - 9)}{(x^2 - 81)(x^2 + 5x + 6)} \\
&= \frac{(x-2)(x-1)\cancel{(x-9)}}{\cancel{(x-9)}(x+9)(x+2)(x+3)} \text{ (factor everything you can, then cancel common factors)} \\
&= \frac{(x-2)(x-1)}{(x+9)(x+2)(x+3)}, \quad x - 9 \neq 0
\end{aligned}$$

Compare to the similar technique from Unit 1: $\frac{5}{46} \times \frac{7}{12} = \frac{5 \times 7}{46 \times 12}$.

It is usually best to leave the final expression in its most factored form, but there will be times when you are solving problems when you will then need to multiply everything out. For now, factored is better!

To divide rational expressions, multiply by the reciprocal of the quantity you are dividing by:

$$\begin{aligned}
\frac{x^2+3x+2}{x^2-9} \div \frac{x-3}{x^2+5x+6} &= \frac{x^2+3x+2}{x^2-9} \cdot \frac{x^2+5x+6}{x-3} \\
&= \frac{(x^2+3x+2)(x^2+5x+6)}{(x^2-9)(x-3)} \\
&= \frac{(x+2)(x+1)(x+2)\cancel{(x+3)}}{\cancel{(x+3)}(x-3)(x-3)} \\
&= \frac{(x+2)(x+1)(x+2)}{(x-3)(x-3)}, \quad x+3 \neq 0 \\
&= \frac{(x+2)^2(x+1)}{(x-3)^2}, \quad x+3 \neq 0
\end{aligned}$$

Compare to the similar technique from Unit 1: $\frac{1}{4} \div \frac{7}{12} = \frac{1}{4} \times \frac{12}{7} = \frac{1 \times 12}{4 \times 7}$.

Example Simplify $\frac{x^2+3x-28}{x^2+14x+49} \div \frac{12-3x^2}{x^2+5x-14}$.

We must factor all the polynomials in the expression:

$$\begin{aligned}
x^2+3x-28 &= (x+7)(x-4) \\
x^2+14x+49 &= (x+7)(x+7) \\
12-3x^2 &= 3(4-x^2) = 3(2-x)(2+x) \\
x^2+5x-14 &= (x-2)(x+7)
\end{aligned}$$

$$\begin{aligned}
\frac{x^2+3x-28}{x^2+14x+49} \div \frac{12-3x^2}{x^2+5x-14} &= \frac{x^2+3x-28}{x^2+14x+49} \times \frac{x^2+5x-14}{12-3x^2} \\
&= \frac{(x^2+3x-28)(x^2+5x-14)}{(x^2+14x+49)(12-3x^2)} \\
&= \frac{\cancel{(x+7)}(x-4)(x-2)\cancel{(x+7)}}{\cancel{(x+7)}\cancel{(x+7)}(3)(2-x)(2+x)} \\
&= \frac{(x-4)(x-2)}{3(2-x)(2+x)}, \quad x+7 \neq 0 \\
&= \frac{(x-4)(x-2)}{-3(x-2)(2+x)}, \quad x+7 \neq 0 \\
&= \frac{(x-4)\cancel{(x-2)}}{-3\cancel{(x-2)}(2+x)} \\
&= \frac{(x-4)}{-3(2+x)}, \quad x+7 \neq 0, x-2 \neq 0 \\
&= -\frac{x-4}{3(2+x)}, \quad x \neq -7, 2
\end{aligned}$$

To add or subtract rational expressions, you must find a common denominator for the expressions. This again requires factoring.

1. Factor each denominator completely.
2. The LCD is a product containing each different factor.
3. If a factor occurs more than once, the LCD will contain that factor repeated the greatest number of times it occurs in any one denominator.

You can use the same process you used in Unit 1 to find the LCD:

$$\boxed{\frac{1}{x^2-2x} - \frac{5}{x^2-4x+4}}$$

Factor — Line them up to get LCD

Factor:

$$x^2-2x = x(x-2) = x \quad (x-2)$$

$$x^2-4x+4 = (x-2)^2 = \quad (x-2)(x-2)$$

$$\text{LCD} = x \quad (x-2)\ (x-2)$$

Therefore,

$$\frac{1}{x^2-2x} - \frac{5}{x^2-4x+4} = \frac{1}{x(x-2)} \cdot \frac{(x-2)}{(x-2)} - \frac{5}{(x-2)(x-2)} \cdot \frac{x}{x}$$

$$= \frac{1(x-2) - 5x}{x(x-2)(x-2)}$$

$$= \frac{x-2-5x}{x(x-2)(x-2)}$$

$$= \frac{-2-4x}{x(x-2)^2}$$

Compare with a similar technique from Unit 1:

$$\boxed{\frac{4}{12} - \frac{5}{18}}$$

Factor:

$$12 = 6 \times 2$$

$$18 = 6 \times \quad 3$$

$$\text{LCD} = 6 \times 2 \times 3 = 36$$

Therefore,

$$\frac{4}{12} - \frac{5}{18} = \frac{4}{12} \cdot \frac{3}{3} - \frac{5}{18} \cdot \frac{2}{2}$$

$$= \frac{4 \cdot 3 - 5 \cdot 2}{36}$$

$$= \frac{12-10}{36} = \frac{2}{36} = \frac{1}{18}$$

Example Simplify $\dfrac{x-8}{x^2-4x+3}+\dfrac{x+2}{x^2-1}$.

Factor everything:

$$\begin{aligned} x^2-4x+3 &= (x-3)(x-1) \\ x^2-1 &= (x-1)(x+1) \end{aligned}$$

Start simplifying:

$$\begin{aligned} \frac{x-8}{x^2-4x+3}+\frac{x+2}{x^2-1} &= \frac{x-8}{(x-3)(x-1)}+\frac{x+2}{(x-1)(x+1)} \\ &= \frac{(x-8)(x+1)}{(x-3)(x-1)(x+1)}+\frac{(x+2)(x-3)}{(x-1)(x+1)(x-3)} \quad \text{(get common denominator)} \\ &= \frac{(x-8)(x+1)+(x+2)(x-3)}{(x-3)(x-1)(x+1)} \quad \text{(simplify)} \\ &= \frac{x^2-7x-8+x^2-x-6}{(x-3)(x-1)(x+1)} \\ &= \frac{2x^2-8x-14}{(x-3)(x-1)(x+1)} \\ &= \frac{2(x^2-4x-7)}{(x-3)(x-1)(x+1)} \quad \text{(numerator is a prime polynomial)} \end{aligned}$$

Example Simplify $\dfrac{x-4}{x^2-4x+3}-\dfrac{x+2}{x^2-1}$.

Factor everything:

$$\begin{aligned} x^2-4x+3 &= (x-3)(x-1) \\ x^2-1 &= (x-1)(x+1) \end{aligned}$$

Start simplifying:

$$\begin{aligned} \frac{x-4}{x^2-4x+3}-\frac{x+2}{x^2-1} &= \frac{x-4}{(x-3)(x-1)}-\frac{x+2}{(x-1)(x+1)} \\ &= \frac{(x-4)(x+1)}{(x-3)(x-1)(x+1)}-\frac{(x+2)(x-3)}{(x-1)(x+1)(x-3)} \quad \text{(get common denominator)} \\ &= \frac{(x-4)(x+1)-(x+2)(x-3)}{(x-3)(x-1)(x+1)} \quad \text{(simplify)} \\ &= \frac{x^2-3x-4-(x^2-x-6)}{(x-3)(x-1)(x+1)} \quad \text{(be careful to distribute minus sign!)} \\ &= \frac{x^2-3x-4-x^2+x+6}{(x-3)(x-1)(x+1)} \\ &= \frac{-2x+2}{(x-3)(x-1)(x+1)} \\ &= \frac{-2\cancel{(x-1)}}{(x-3)\cancel{(x-1)}(x+1)} \\ &= \frac{-2}{(x-3)(x+1)}, \quad x \neq 1 \end{aligned}$$

An Application We can express the average velocity of the object over a time interval by the formula:

$$\text{Average Velocity from } t = t_1 \text{ to } t = t_2 \;= \frac{\text{distance traveled}}{\text{change in time}}$$

If we drop a ball from rest (meaning initial velocity is zero) at a height h meters above ground, then the height above ground S is approximated by the following equation:

$$S = -5t^2 + h,$$

$$\begin{aligned} \text{where } h = &\ \text{height above ground from which the object is dropped in meters} \\ t = &\ \text{time in seconds} \end{aligned}$$

The average velocity of the ball during time interval from t_1 to t_2 is given by

$$\begin{aligned} \text{Average Velocity from } t = t_1 \text{ to } t = t_2 \;&= \frac{\text{distance traveled}}{\text{change in time}} \\ &= \frac{(-5t_2^2 + h) - (-5t_1^2 + h)}{t_2 - t_1} \\ &= \frac{-5t_2^2 + h + 5t_1^2 - h}{t_2 - t_1} \\ &= \frac{-5(t_2^2 - t_1^2)}{t_2 - t_1} \\ &= \frac{-5(t_2 + t_1)(t_2 - t_1)}{t_2 - t_1} \\ &= -5(t_2 + t_1) \text{ meters/second}, \quad t_2 \neq t_1 \end{aligned}$$

Notice that factoring and simplifying lead to a much simpler expression for the average velocity in the end. The result is negative since the ball is moving downwards, towards the earth. To treat the case when $t_2 = t_1$ requires calculus.

Example You are standing on a cliff overlooking the ocean. You are 180 meters above the ocean. You drop a pebble into the ocean (dropping implies the initial velocity is 0 m/s). How long will it take for the pebble to hit the water? What is the velocity average velocity of the pebble in the second after you drop it? In the second before it hits the water?

Solution: The pebble hits the water when $S = 0$ and $h = 180$ m, so we need to solve

$$\begin{aligned} 0 &= -5t^2 + 180 \\ 0 &= -5(t^2 - 36) && \text{(common factors in terms)} \\ \frac{0}{-5} &= \frac{-5(t^2 - 36)}{-5} && \text{(divide each side by } -5\text{)} \\ 0 &= t^2 - 36 && \text{(difference of squares)} \\ 0 &= (t-6)(t+6) \\ (t-6) = 0 &\text{ or } (t+6) = 0 && \text{(zero factor property)} \\ t = 6 &\text{ or } t = -6 && \text{(simplify)} \end{aligned}$$

Exclude the $t = -6$ as unphysical, and we see the pebble hits the water after 6 seconds.

$$\begin{aligned} \text{Average Velocity from } t = 0 \text{ to } t = 1 \;&= \frac{\text{distance traveled}}{\text{change in time}} \\ &= -5(t_2 + t_1) = -5(1 + 0) = -5 \text{ meters/second} \\ \text{Average Velocity from } t = 5 \text{ to } t = 6 \;&= \frac{\text{distance traveled}}{\text{change in time}} \\ &= -5(t_2 + t_1) = -5(11) = -55 \text{ meters/second} \end{aligned}$$

So we see that the pebble sped up dramatically as it fell.