Growth of Savings: Simple Interest

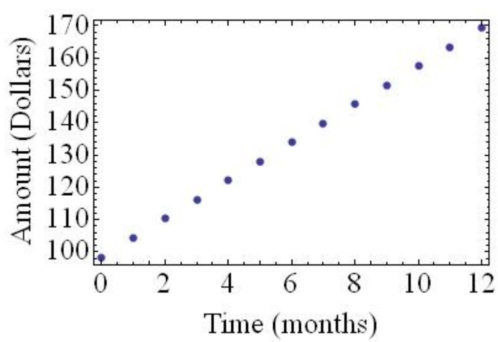
Simple interest pays interest only on the principal, not on any interest which has accumulated. Simple interest is rarely used for saving accounts, but it is used for bonds.

Example You put \$98.45 in a savings account which pays simple interest of 6% a month. How much money do you have in the savings account after 4 months?
Solution To answer this question, we can build from what we know. Simple interest means we pay interest only on the initial amount deposited (principal), which was \$98.45. The interest amount will be 6%= 6/100 = 0.06 of the principal, and added to the account balance once a month.

Interest Period	Date	Interest Added	Accumulated Amount
0	Jan 1	0	\$98.45
1	Feb 1	\$98.45×0.06=\$5.91	\$98.45 + \$5.91 = \$104.36
2	Mar 1	\$98.45×0.06=\$5.91	\$104.36 + \$5.91 = \$110.26
3	Apr 1	\$98.45×0.06=\$5.91	\$110.26 + \$5.91 = \$116.17
4	May 1	\$98.45×0.06=\$5.91	\$116.17 + \$5.91 = \$122.08

This table is the form an Excel spreadsheet would take to calculate simple interest. Notice the first row is an initialization and it is the second row that contains formulas.

We see that the growth is by a constant amount (\$98.45×0.06=\$5.91) every time period (month in this case). This is the requirement for linear or arithmetic growth. It gets the name linear since the graph of the amount versus the time is a straight line (linear function).



Simple Interest Formula

For simple interest of r percent paid every time period with a principal P, we get

Years	Accumulated Amount
0	P
1	$P + Pr = P(1 + r)$
2	
3	
4	
	$\vdots$
t	

ie., for a principal of P with simple interest of $r\%$ paid every time period, we get an accumulated amount after t years of

$\Rightarrow$

The formula gives you another way of calculating a quantity that could be done using a spreadsheet style table.

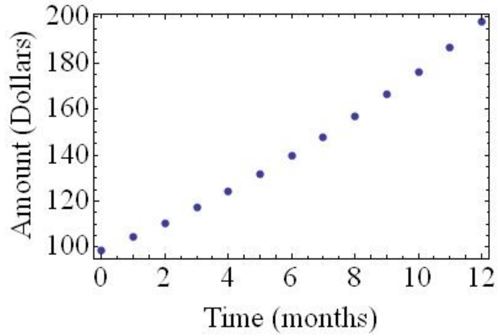
Growth of Savings: Compound Interest

Compound interest pays interest on the principal and the accumulated interest, not just the principal.

Example You put \$98.45 in a savings account which pays compound interest of 6% a month. How much money do you have in the savings account after 4 months?
Solution To answer this question, we can build from what we know. Compound interest means we pay interest on the accumulated amount in the account. The interest amount will be $6\% = 6/100 = 0.06$ of this amount, and added to the account balance once a month.

Compounding Period	Date	Interest Added	Accumulated Amount
0	Jan 1	0	\$98.45
1	Feb 1	\$98.45×0.06=\$5.91	\$98.45 + \$5.91 = \$104.36
2	Mar 1	\$104.36×0.06=\$6.26	\$104.36 + \$6.26 = \$110.62
3	Apr 1	\$110.62×0.06=\$6.64	\$110.62 + \$6.64 = \$117.26
4	May 1	\$117.26×0.06=\$7.04	\$117.26 + \$7.04 = \$124.29

We see that the amount of growth increases as time increases. The amount of growth is proportional to the amount present, which is the requirement for geometric growth.



Interest Terminology

Savings problems typically involve a bit more terminology than we've used so far.

The compounding period (or just period) is the time which elapses before compound interest is paid.

The time when compounding is done effects the accumulated amount, since the current amount affects the amount of interest added, and the current amount will change if we compound more frequently.

The nominal rate is the stated rate of interest for a specified length of time. The nominal rate does not take into account how interest is compounded!

The effective rate is the actual percentage rate of increase for a length of time which takes into account compounding. It represents the amount of simple interest that would yield exactly as much interest over that length of time.

The effective annual rate (EAR) is the effective rate given over a year. For savings accounts, the EAR is also called the annual percentage yield (APY).

Compound Interest Formula

For a nominal annual rate r, compounded m times per year, we have $i = r/m$ as the interest rate per compounding period. Now let's try to derive a formula for compound interest.

Compounding Period	Amount
0	P
1	$P + Pi = P(1 + i)$
2	
3	
4	
⋮	
n	

ie., for a principal of P with compound interest of $i = r/m$ paid every compounding period, we get an accumulated amount after $n = mt$ compounding periods (t is number of years, m is number of compounding periods per year) of

$\Rightarrow$

Annual Percentage Yield (APY)

By definition, the APY is the simple interest rate that earns the same interest as the compound interest after one year ($t = 1$).

$$\begin{aligned} \text{Compound Interest } A &= P\left(1+\frac{r}{m}\right)^{mt} = P\left(1+\frac{r}{m}\right)^{m} \\ \text{Simple Interest } A &= P\left(1+rt\right) = P\left(1+\text{APY}\right) \end{aligned}$$

Set these quantities equal, and solve for APY:

$\Rightarrow$

A Limit to Compounding

Sketch the graph of the accumulated amount for 10 years if the principal is $P =$\$1000 and the annual interest rate is $r = 10\%$ for simple interest, compound interest compounded yearly, compound interest compounded quarterly, and compound interest compounded daily (assume 365 days in a year).

To get the values, we can use the formulas we derived. Here is the process for getting the accumulated amount after 1 year (so $t = 1$ in all formulas); the rest are calculated in a similar fashion using $t = 2, 3, 4, \ldots$.

Simple interest after 1 year:

$$A = P(1 + rt) = \$1000(1 + 0.10 \times 1) = \$1100.00 \text{ after 1 year.}$$

Compound interest compounded yearly ($m = 1$, $i = r/m = 0.10/1 = 0.10$, and $n = mt = 1$):

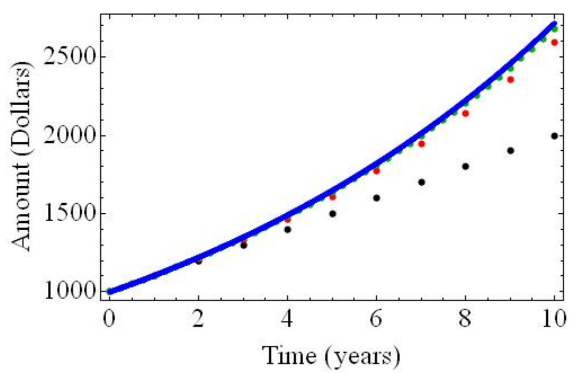
$$A = P(1 + i)^n = \$1000(1 + 0.10)^1 = \$1100.00 \text{ after 1 year.}$$

Compound interest compounded quarterly ($m = 4$, $i = r/m = 0.10/4 = 0.025$, and $n = mt = 4$):

$$A = P(1 + i)^n = \$1000(1 + 0.025)^4 = \$1103.81 \text{ after 1 year.}$$

Compound interest compounded daily ($m = 365$, $i = r/m = 0.10/365 = 0.000273973$, and $n = mt = 365$):

$$A = P(1 + i)^n = \$1000(1 + 0.000273973)^{365} = \$1105.16 \text{ after 1 year.}$$



black: simple interest.
red: compound interest, compounded yearly.
green: compound interest, compounded quarterly.
blue: compound interest, compounded daily.

- The curves are all essentially the same for short times.
- There are more points for compounding quarterly than yearly since interest is paid more often during the year.
- There is not much difference over 10 years to compounding quarterly and compounding daily.
- Compounding more frequently leads to a larger accumulated balance, but there is a limit to this process. The limit would be if we compounded continuously.

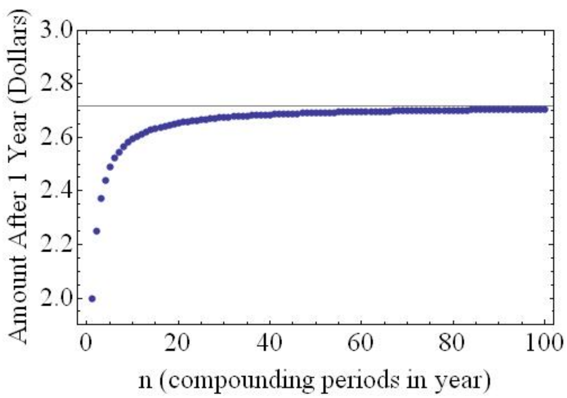
Compounding Continuously

Consider a principal $P = \$1$ and a rate of r=100% which is compounded over shorter and shorter time periods. We are interested in how much the accumulated amount will be after one year.

Compound interest compounded n times a year ($i = 1/m$, and $n = mt = m$ (to get one year, $t = 1$)):

$$A = P(1+i)^n = \left(1 + \frac{1}{m}\right)^m \text{ after 1 year.}$$

There is a table in the text of these numbers, here is a sketch



We see that the accumulated amount is approaching a number:

$$\left(1 + \frac{1}{m}\right)^m \sim 2.71828... \text{ if } m \text{ is very large.}$$

This number is similar to $\pi = 3.14...$ in that it is mathematically significant and appears in many situations, and so we give it a special designation:

$$e \sim \left(1 + \frac{1}{m}\right)^m \sim 2.71828... \text{ if } m \text{ is very large.}$$

This leads the the continuous interest formula, which is

$A = Pe^{rt}$ after t years if interest is compounded continuously at annual rate r.

The function e^{rt} is called the exponential function. The continuous interest formula is the upper limit on the accumulated amount that can accrue due to compounding interest.

Review of interest formulas (P principal; A accumulated amount; r annual rate)

- Simple interest: $A = P(1 + rt)$ is the amount after t years.
- Interest rate per compounding period $i = r/m$; m compounding periods per year
- Compound interest, after n compounding periods: $A = P(1 + i)^n$.
- Continuously compounded interest: $A = Pe^{rt}$ is the amount after t years.
- $\text{APY} = \left(1 + \frac{r}{m}\right)^m - 1$ is the annual percentage yield.

The formulas allow us to answer questions which would be difficult to answer using a table, and also to answer questions quickly without a lot of calculation. However, the tables allow us to answer questions that do not match the conditions under which the formulas were derived. Therefore, both formulas and spreadsheet tables are useful (and necessary) in understanding how personal finance works.

Geometric Series

Consider the following series: $1 + x + x^2 + x^3 + \cdots + x^{n-1}$.

We need to figure out a way to write this without the $\cdots$. Here's how:

$$\begin{aligned} s &= 1 + x + x^2 + x^3 + \cdots + x^{n-1} \\ \text{subtract } \underline{xs} &= \underline{\quad x + x^2 + x^3 + \cdots + x^{n-1} + x^n} \\ s - sx &= 1 - 0 - 0 - 0 - \cdots - 0 - x^n \\ s - sx &= 1 - x^n \end{aligned}$$

Now solve for s:

$\Rightarrow$

Therefore, a <u>geometric series</u> has the following sum:

$\Rightarrow$

Exponential and Natural Logarithms

As we have seen, some of our equations involve exponents. To effectively deal with exponents, we need to be able to work with exponentials and logarithms.

The exponential and natural logarithm functions are inverse functions, and related by the following rules

$$\begin{aligned} e &= \left(1 + \frac{1}{m}\right)^m \text{ for } m \text{ large} \\ e &= 2.71828\ldots \\ \ln(e^A) &= A \text{ where } A \text{ is a constant} \\ e^{\ln(A)} &= A \end{aligned}$$

If the base of the exponent is not e, we can use the following rule:

$$\begin{aligned} \ln(b^A) &= A\ln(b) \text{ where } A \text{ and } b \text{ are constants} \\ \ln(2^A) &= A\ln(2) \\ \ln((1+r)^A) &= A\ln(1+r) \end{aligned}$$

Note:

- The natural logarithm acts on a number, so we read $\ln(2)$ as "The natural logarithm of 2".
- We would never write $\ln(2) = 2\ln$, since this loses the fact that the natural logarithm must act on something. It is a functional evaluation, not a multiplication.
- This is similar to trigonometric functions, like $\sin(\pi)$. The sine function is being evaluated at π when we write $\sin(\pi)$, just as the natural logarithmic function is being evaluated at 2 when we write $\ln(2)$.

Example Solve the equation $(1.002)^k = 1.0832$ for k.
Solution We will need to use the natural logarithm here:

$$\begin{aligned} 1.002^k &= 1.0832 \\ \ln(1.002^k) &= \ln(1.0832) \\ k\ln(1.002) &= \ln(1.0832) \\ k &= \frac{\ln(1.0832)}{\ln(1.002)} = 39.9998 \sim 40 \end{aligned}$$

Accumulation

An important aspect of saving is the idea of accumulation, which answers the question: *What size deposit do I have to make at regular time interval d to save a certain amount of money in a certain amount of time?* This would be important for saving for retirement, or a down payment on a house, or a car, or a child's education.

Obviously, if there was no interest, you would just break the amount you need to save into d even pieces and deposit that amount regularly. Interest makes the problem more interesting!

Example You begin saving for retirement at age 35 by paying \$100 a month into an account paying 6% annual interest compounded monthly. How much will you have in savings by the time you are 65?

The easiest way to think of this is backwards, starting by what happens at age 65. For interest compounded monthly at an annual rate of 6%, we have $i = r/m = 0.06/12 = 0.005$.

The last deposit you make will be \$100, and earn no interest (or interest for 0 months): \$100

The penultimate deposit will be \$100, and will earn interest for 1 month: $\$100(1+i)^1$.

The second last deposit will be \$100, and will earn interest for 2 month: $\$100(1+i)^2$.

This process continues, right up until the first deposit is made. In $65 - 35 = 30$ years, you will make $30 \times 12 = 360$ monthly deposits.

The amount you save is

$$\begin{aligned} A &= \$100 + \$100(1+i)^1 + \$100(1+i)^2 + \cdots + \$100(1+i)^{359} \\ &= \$100\left[1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{359}\right] \end{aligned}$$

We stop at 359 since we started at 0, not 1.

This is a geometric series, with $x = (1+i)$ and $n = 360$.

Therefore, we can write

$$A = \$100\left[1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{359}\right] = \$100\left[\frac{(1+i)^{360} - 1}{(1+i) - 1}\right] = \$100\left[\frac{(1+i)^{360} - 1}{i}\right].$$

The amount we will save by the age of 65 is

$$A = \$100\left[\frac{(1+i)^{360}-1}{i}\right] = \$100\left[\frac{(1+0.005)^{360}-1}{0.005}\right] = \$100\,451.50.$$

Only \$36, 000 of this is due to the deposits. The rest is interest.

We now have a new formula: for a uniform deposit d per compounding period and an interest rate of i per period, the amount A accumulated after n periods is given by the savings formula:

$\Rightarrow$

Example What should your monthly deposit be in a savings account with 7% annual interest compounded monthly if you want to save \$3000 in 24 months for the down payment on a new car?

Example You wish to remodel your kitchen, and estimate it will cost \$35,000 to do. If you can afford to save \$500 a month in a savings account that earns 4% annual interest, how long will it take you to save enough to remodel the kitchen?

Exponential Decay

Exponential Decay is geometric growth with a negative rate of growth.

If $i > 0$, then

growth: $A = P(1 + i)^n$ A is the accumulated amount
decay: $V = P(1 - i)^n$ V is the value

This decrease in the amount models inflation over a short time period, where the value of the dollar goes down geometrically, or depreciation, where the value of an item decreases.

You can think of the value of a dollar as depreciating over time much like the value of an item depreciates over time (cars are a prime example of an item whose value decreases over time).

The actual price of an item at any time is said to be in current dollars. To compare prices of items from different times (which will take into account inflation), we use constant dollars, which are dollars from a particular year.

Example Suppose you bought a car in early 2007 for $10,000. If its value (in current dollars) depreciates steadily at 12% per year (cars typically depreciate at 15-20% a year), what will be its value (in current dollars) in early 2010?

The Consumer Price Index

If inflation stayed constant over the years, we could use the above ideas to compare the cost of an item in an earlier year with the cost of the item today. However, inflation is not a constant! It varies over time.

The Consumer Price Index (CPI) `http://www.bls.gov/cpi/home.htm` allows us to compare the cost of items in different years.

The CPI represents costs of a basket of goods (food, housing, transportation, etc). This cost is measured each year for the same set of goods. The cost will vary over time, and also over region.

There must be some base number against which all the other numbers are compared, so the CPI for the years 1982–1984 is set to 100 (this is arbitrarily chosen by the Bureau of Labor, they could have chosen something else).

If you want to relate the cost of two items in different years you use the relation:

$$\frac{\text{Cost in Year } A}{\text{Cost in Year } B} = \frac{\text{CPI in Year } A}{\text{CPI in Year } B}$$

There are different CPI for different regions or metro areas, and also for different sets of goods.

The FAQ on the CPI website contains a wealth of information about how to use the CPI effectively.

Example What is the value of a dollar from 1970 in 1987 dollars?

The online inflation calculator: `http://www.bls.gov/data/inflation_calculator.htm` agrees–it shows that $1 in 1987 has the same buying power as 34 cents in 1970. The online calculator just looks up the data and uses the ratio we did.

Example When buying a new home, Sam learns from her parents that they paid $39,000 in 1967 for the house she grew up in. Seeing that houses cost much more today, Sam tells her parents that they got an incredibly good deal on their house, and she is spending much more on her $150,000 house today.
Her parents chuckle, and tell Sam that the cost of items has gone up over the years, and if she really wants to compare the cost of her childhood home to her new house, she needs to take that into account.
How much would Sam's childhood home be in 2004 dollars?