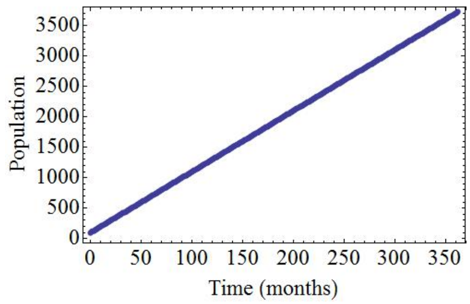
Introduction

Populations grow and decrease over time, and there are different mathematical models for this. Some common models are the following–there are many others!

Arithmetic Growth

Here, the growth is by a fixed amount each time period. So we have

$$\text{growth rate } = P' = r = \text{ constant.}$$

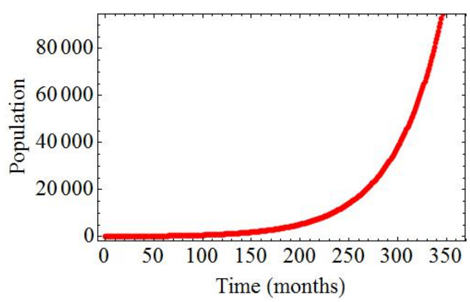


This is the same as simple interest, and leads to a straight line for the population, since the slope of the population line (rise/run) is always the same.

Geometric Growth

Here, the growth is by a fixed percentage of the current population each time period. So we have

$$\text{growth rate } = P' = rP.$$

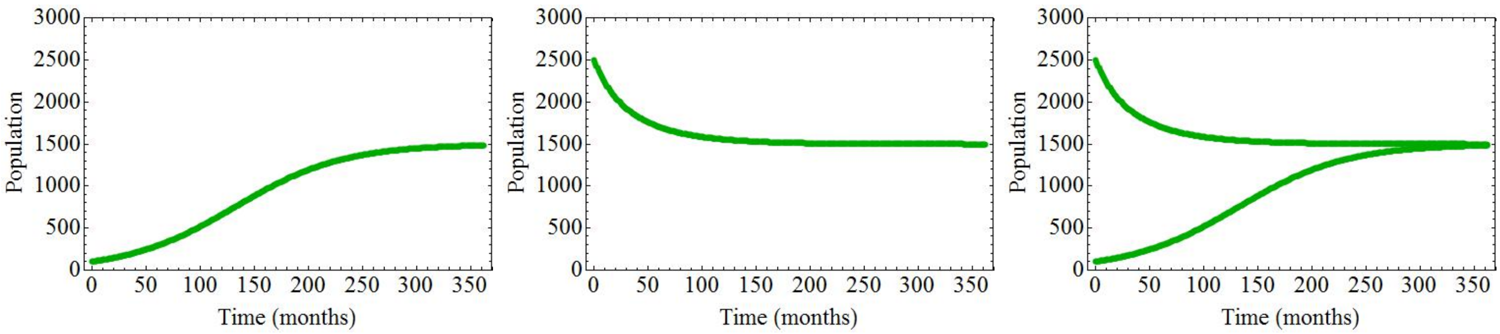


This is the same as compound interest, and leads to a curving line for the population, since the slope of the population line (rise/run) is always increasing.

Logistic Growth

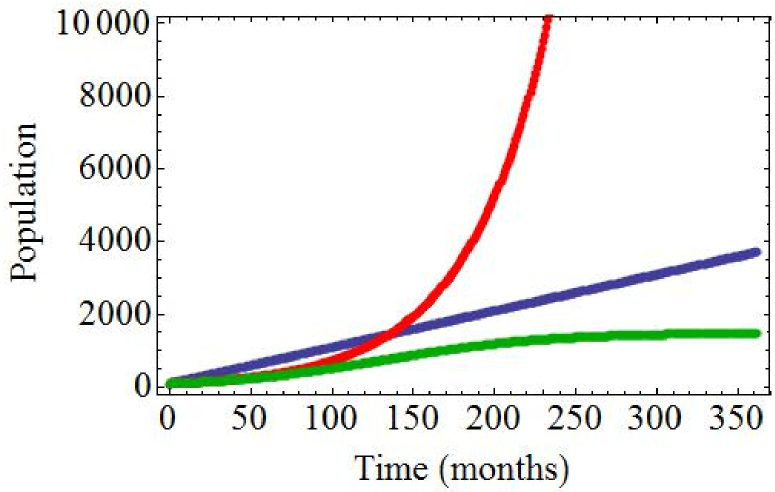
Both the previous growth models have a flaw–they would indicate that the population can grow without restraint, becoming very large. But that isn't what happens in nature–the population is constrained by the resources available to it and often there is a maximum population that can be supported. This maximum population is called the carrying capacity M, and logistic growth incorporates this into the growth rate.

$$\text{growth rate } = P' = rP\left(1 - \frac{P}{M}\right).$$



For logistic growth, the growth is initially increasing, then eventually decreases and reaches a maximum at M. If the initial population is above M, the population will decrease to the carrying capacity.

Comparing these different models, we see that they are all very similar for short times, but the long time behaviour is quite different.



More advanced models will also contain the idea of extinction, and if a population falls below a certain amount they will eventually become extinct.

The Rule of 70: Population Doubling

For estimating how long it takes for a population to double, one can use the Rule of 70:

> If a country's population continues to grow at a constant rate of $r\%$ per year, then it will double in size every $70/r$ years.

Since population growth and savings/borrowing are closely related, you should not be surprised to learn that there is a similar rule (The Rule of 72) for estimating an investments doubling time.

This rule arises from the geometric growth model, with $i = r/m = r$ since the populations are measured each year. Since geometric growth is really just compound interest, we have

$$\begin{aligned}
\text{New Population} &= \text{Old Population}(1+i)^n \\
2\times\ \text{Old Population} &= \text{Old Population}(1+r)^n \\
2 &= (1+r)^n \\
\ln(2) &= \ln((1+r)^n) \\
\ln(2) &= n\ln(1+r) \\
n &= \frac{\ln(2)}{\ln(1+r)}
\end{aligned}$$

For r a small number, $\ln(1+r) \sim r$ (check it out on a calculator), and $\ln 2 = 0.693 \sim 0.70$. So we have

$$n \sim \frac{0.70}{r} = \frac{70}{r\%} \text{ where } r \text{ is the percentage.}$$

as the time it takes for the population to double.

Example Exercise 6 Using the rule of 70, when will the populations of the following countries have doubled?

(a) Africa, 925 Million, growth rate of 2.4% in 2007.
(b) USA, 301 million, growth rate of 0.8% in 2007.

Solution:

$$\begin{aligned}
\text{Africa time to double} &= \frac{70}{r} = \frac{70}{2.4} = 29 \text{ years.} \\
\text{USA time to double} &= \frac{70}{r} = \frac{70}{0.8} = 87.5 \text{ years.}
\end{aligned}$$

Example Wisconsin's electricity demand in 1991 was about half from business use and half from residential use, and increased at 2.8% per year. When will the demand be twice what it was in 1991?

$$\text{time to double} = \frac{70}{r} = \frac{70}{2.8} = 25 \text{ years.} \qquad 1991 + 25 = 2016.$$

How Long will Nonrenewable Resources Last

Resources come in two basic types: renewable resources and nonrenewable resources. Once nonrenewable resources are used up, it takes a long time for the ecosystem to generate more, if more can be generated at all. Gasoline and coal are examples of nonrenewable resources.

How long nonrenewable resources last is obviously a very important question, and so modeling this situation is critical. We have two basic types of models:

- If the rate of use of the resource is constant, then we can calculate the static reserve.
- If the rate of use is increasing (which typically happens as the population increases), then a better model would be to calculate the exponential reserve

Static Reserve

If we use U units per year and have a reserve of S units of the resource, then we have enough resources to last $\frac{S}{U}$ years.

Exponential Reserve

If we use U_1 units in the first year, and the usage increases at $r\%$ each year, and have a reserve of S units of the resource, then the problem is more complex. But it is still just a compound interest problem at heart, so we can answer it. We can modify the savings formula, with $i = r/m = r$ to answer the question of when we run out of the resource.

$$\begin{aligned} A &= d\left[\frac{(1+i)^n - 1}{i}\right] \\ S &= U\left[\frac{(1+r)^n - 1}{r}\right] \end{aligned}$$

We need to solve this for n, the time it takes to deplete the resource.

$$\begin{aligned} \frac{(1+r)^n - 1}{r} &= \frac{S}{U} \\ (1+r)^n - 1 &= \frac{S}{U}r \\ (1+r)^n &= 1 + \frac{S}{U}r \\ \ln((1+r)^n) &= \ln\left(1 + \frac{S}{U}r\right) \\ n\ln(1+r) &= \ln\left(1 + \frac{S}{U}r\right) \\ n &= \frac{\ln\left(1 + \frac{S}{U}r\right)}{\ln(1+r)} \end{aligned}$$

This gives the number of years that it will take to deplete the resource.

Example In 2004, the natural gas resources totaled 6076.5 trillion cubic feet, while annual consumption was 90 trillion cubic feet per year. World consumption is projected to increase by 2.2% per year through 2025.

(a) What is the static reserve in 2004?

(b) What is the exponential reserve in 2004?

Solutions:

(a) Static reserve $= S/U = 6076.5/90 = 67.5$ years

(b) Exponential reserve $= \dfrac{\ln\left(1+\frac{S}{U}r\right)}{\ln(1+r)} = \dfrac{\ln\left(1+\frac{6076.5}{90}0.022\right)}{\ln(1.022)} = 41.8$ years.

Sustaining Renewable Resources

A renewable resource is one that replenishes itself (any form of wildlife, trees, bamboo, corn, etc). The question with renewable resources is not when will they run out, but how much can we harvest and still leave enough of the population to regenerate itself. This is particularly important for wild populations of animals which are harvested–especially fish.

Reproduction Curves

A reproduction curve is a mathematical model that gives the population size in the next year if the population size this year is known:

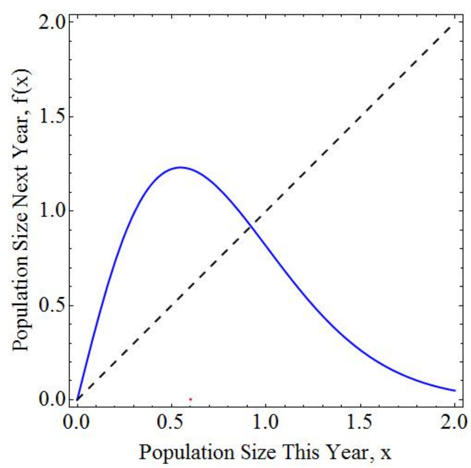


Figure 1: Population curve that relates the population in a current generation x to the population in the next generation by the formula $y = f(x)$. The dashed line is $y = x$ which represents no population change from year to year.

The basic shape of this curve is no accident. It rises on the left since there are enough resources for the population to increase, and falls on the right since the population is too large for the available resources and must decline.

The details of the shape of the reproduction curve depend on the particular population being studied, but the basic shape is always the same.

The reproduction curve is usually used in the following manner. Vertical lines represent moving from one year to the next. The amount of the vertical line between the line $y = x$ and the reproduction curve represent the natural increase or natural decrease of the population from one year to the next. An equilibrium population is a population which does not change from year to year. An equilibrium population occurs where the reproduction curve intersects the line $y = x$.

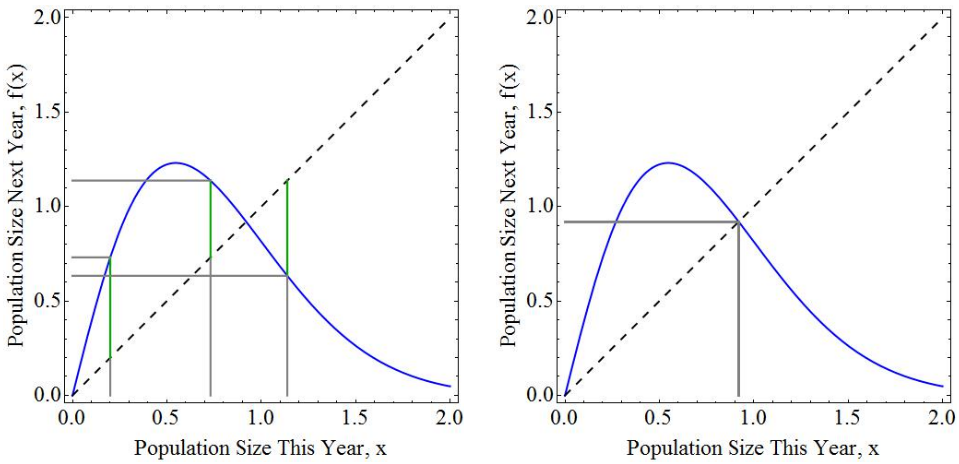


Figure 2: Left: Population starts at 0.2. In following years, the population is 0.725, 1.1.25, 0.625. The green lines represent population growth (above $y = x$) or decrease (below $y = x$) from year to year. Right: Equilibrium population.

Sustained Yield Harvesting of a Population

A simple model of harvesting a population would include the idea that greater effort would result in greater harvest. The harvest line could therefore be a straight line with steeper slope indicating greater harvest. It looks like the $y = x$ line we used, but it has a different slope to indicate different levels of harvesting.

A sustained yield harvesting policy is a policy that if continued indefinitely will maintain the same yield. We are most interested in the maximum sustainable yield, which would maintain the population at at a constant fixed amount, and harvest the maximum each year (if we harvest less than the maximum, the population would increase from year-to-year). This is described in Fig. 3.

An Alternate Way of Showing The Population Dynamics: Cobweb Diagrams

Another way of presenting this information is shown in Fig. 4. The horizontal lines that move towards the straight line $y = x$ are used to "reset" to the next year (and therefore represent the natural increase or natural decrease of the population, since they are the same length as the green lines on the left).

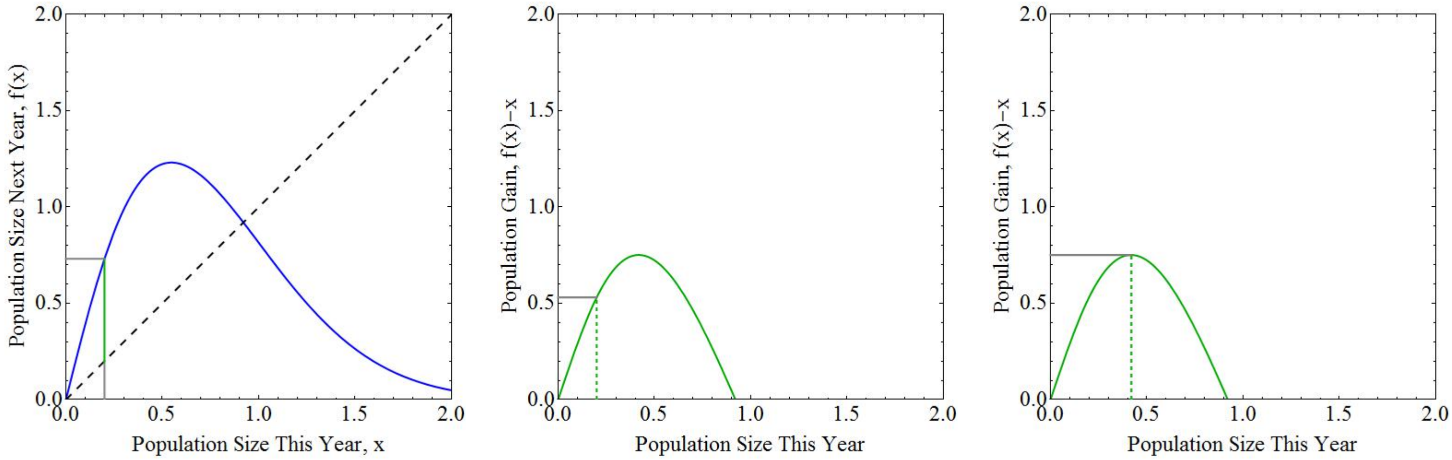


Figure 3: Left: Population curve. Center: Population Gain curve, $f(x) - x$. The dashed line shows the population gain going from 0.2 to 0.725, which is $0.725 - 0.2 = 0.525$. Right: The maximum sustainable yield would maintain the population at 0.42 year-to-year, with a maximum sustainable yield of 0.75.

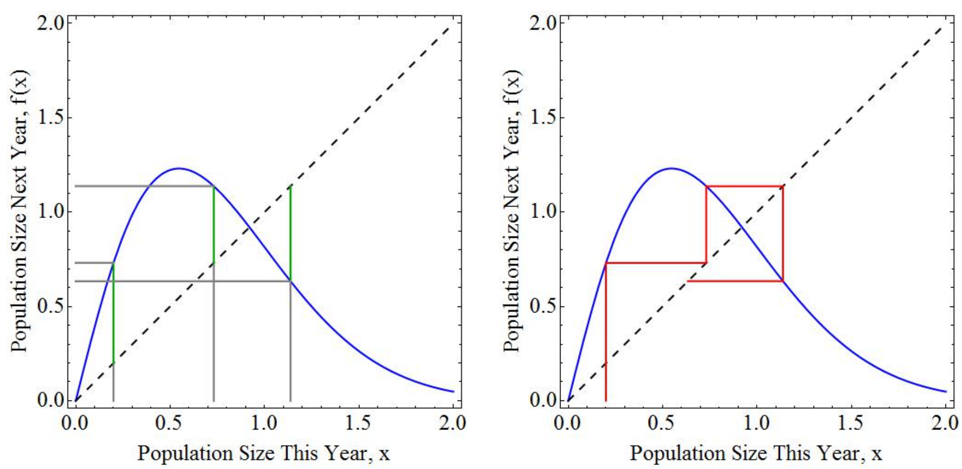
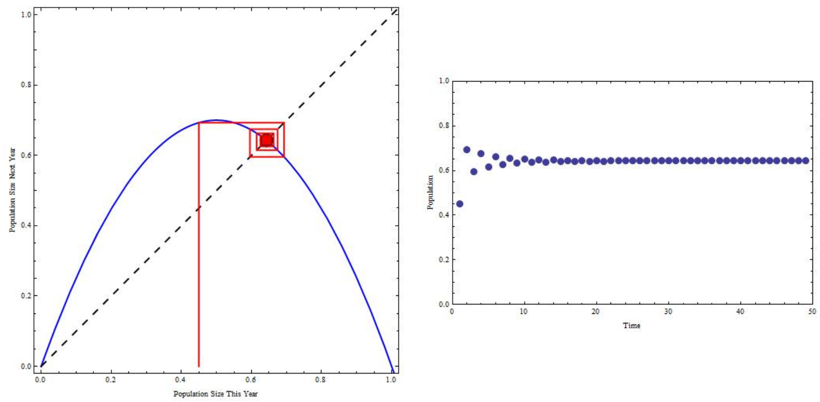
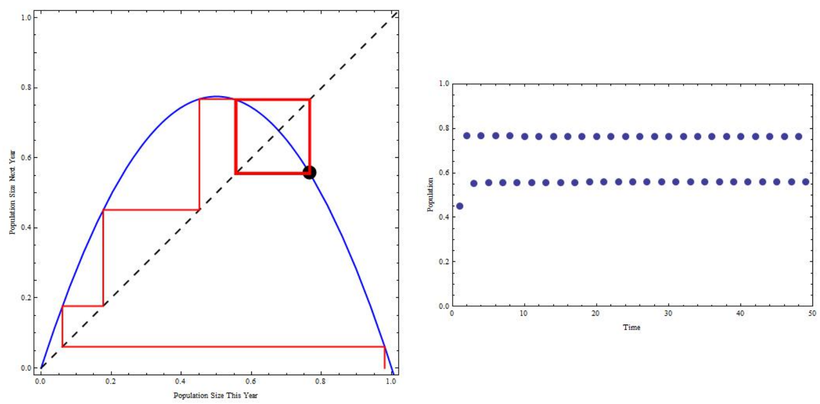


Figure 4: Left: Population curve. Right: Cobweb diagram.

Chaos in Biological Populations

One definition of chaos is that knowing the current population does not allow you to predict the future population. Of course, mathematically we can predict future populations, so a better definition of chaos would be a system for which small changes in initial conditions lead to large differences later. This is really what predictability is all about–the predictions for a system without chaos would be roughly the same if we changed our starting position by a small amount. This isn't true for chaotic systems, and much of nature is chaotic.


Population Size Next Year
Population Size This Year
Population
Time


Population Size Next Year
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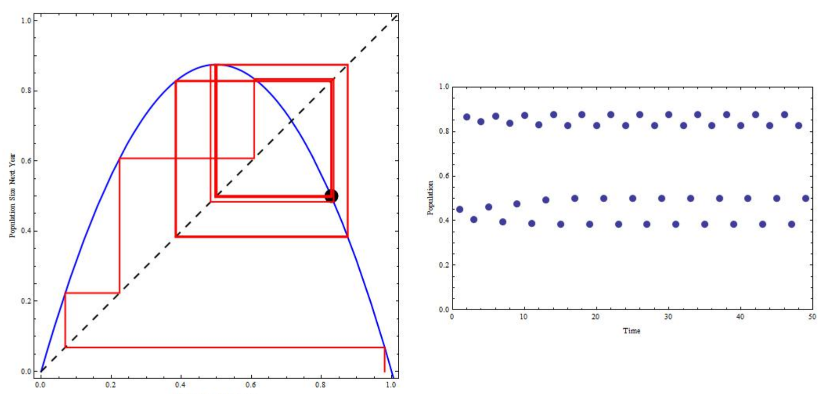
Population Size Next Year
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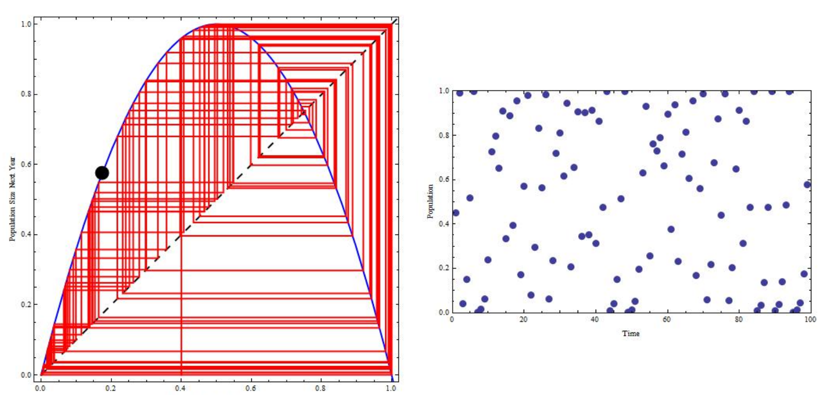
Population
Time


Population Size Next Year
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