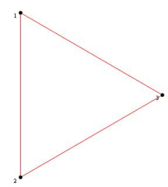
Chapter 2: Business Efficiency

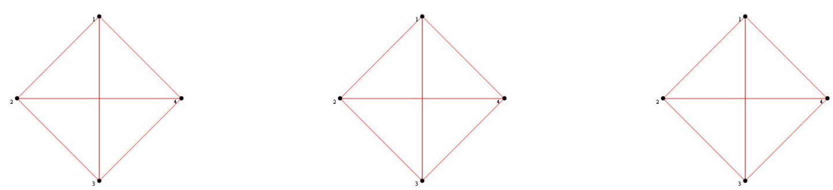
Hamiltonian Circuits

The total number of distinct Hamiltonian circuits on complete graph of $n > 2$ vertices is $\dfrac{(n-1)!}{2}$.

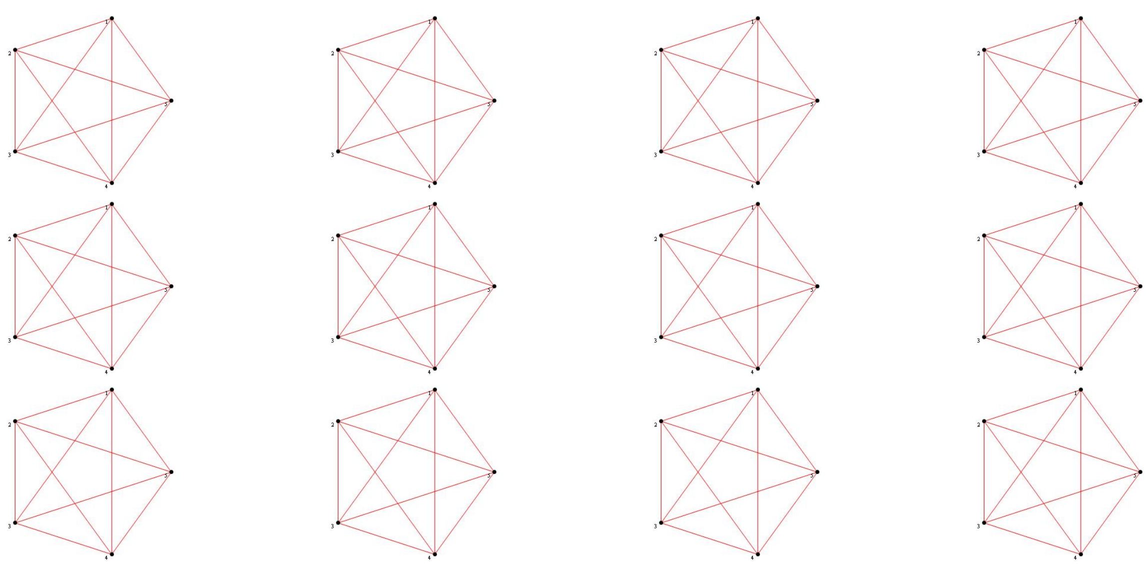
$n = 3$ should have $\dfrac{(3-1)!}{2} = \dfrac{2!}{2} = \dfrac{2 \times 1}{2} = 1$ distinct Hamiltonian circuit.



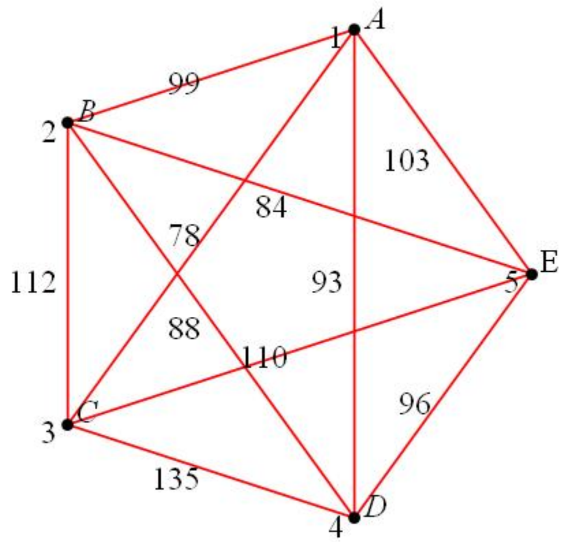
$n = 4$ should have $\dfrac{(4-1)!}{2} = \dfrac{3!}{2} = \dfrac{3 \times 2 \times 1}{2} = 3$ distinct Hamiltonian circuits.



$n = 5$ should have $\dfrac{(5-1)!}{2} = \dfrac{4!}{2} = \dfrac{4 \times 3 \times 2 \times 1}{2} = 12$ distinct Hamiltonian circuits.



Method of Trees For example, if we wanted to drive between 5 cities, the cost could be the distance between the cities.



Finding a minimum-cost Hamiltonian circuit for the above situation would correspond to optimizing (minimizing in this case) the distance travelled to visit the five cities.

There are several methods for trying to find a minimum-cost Hamiltonian circuit. We will look at three: a brute force method based on the method of trees, The Nearest Neighbour Algorithm, The Sorted Edges Algorithm.

Brute-Force Method: The Method of Trees

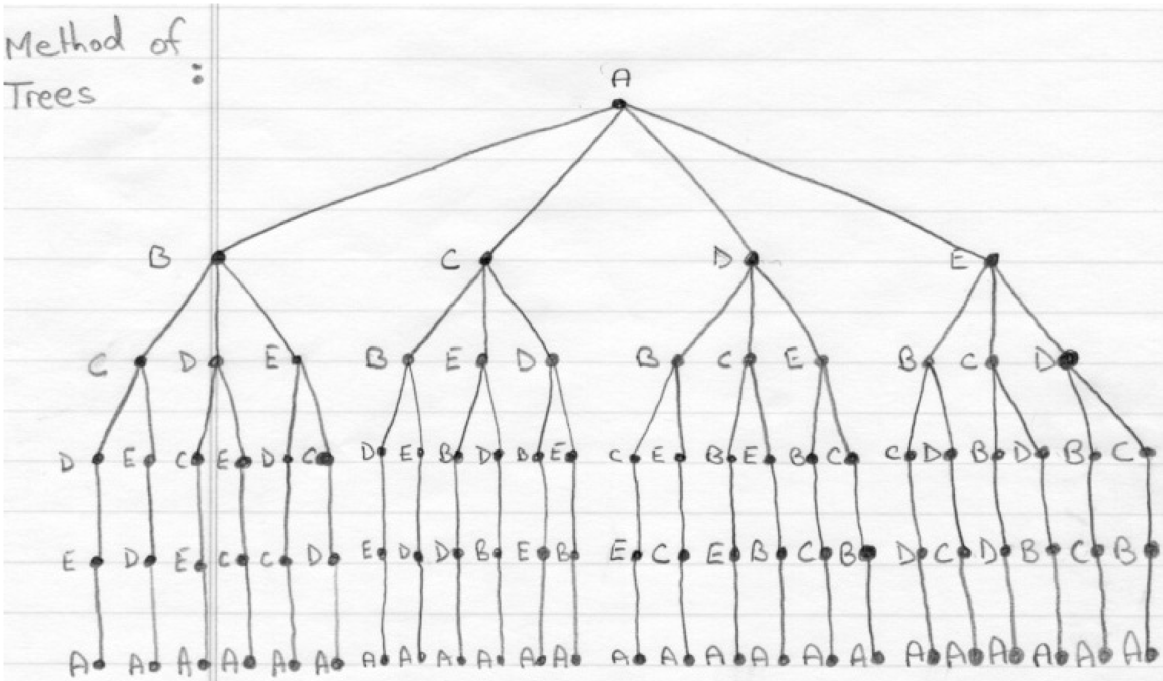
- generate all the possible Hamiltonian circuits, using the Method of Trees
- determine the total distance travelled for each tour, and
- choose the one with minimum distance. This is the minimum cost Hamiltonian circuit.

For a complete graph on 5 vertices, the method of trees is as follows:

We choose a starting vertex.
Then choose from any of the remaining 4 vertices.
Then choose from any of the remaining 3 vertices not already visited.
Then choose from any of the remaining 2 vertices not already visited.
Then choose the remaining 1 vertex not already visited.
Then return to starting vertex.

We can get $4 \cdot 3 \cdot 2 \cdot 1 = 24$ circuits using the above method.

Pictorially, this looks like the following:



The circuits are duplicated, since the method of trees finds the circuit travelled in one direction and then again travelled in the other, i.e., ABCDEA and AEDCBA are the same Hamiltonian circuit.

Therefore, we have 12 different Hamiltonian circuits for a complete graph on 5 vertices.

We have found all the Hamiltonian circuits using the method of trees.

Now, we find the distance of each Hamiltonian circuit:

Hamiltonian Circuit	Cost (Distance Travelled)
ABCDEA	$99 + 112 + 135 + 96 + 103 = 545$
ABCEDA	$99 + 112 + 110 + 96 + 93 = 510$
ABDCEA	$99 + 88 + 135 + 110 + 103 = 535$
ABDECA	$99 + 88 + 96 + 110 + 78 = 471$
ABEDCA	$99 + 84 + 96 + 135 + 78 = 492$
ABECDA	$99 + 84 + 110 + 135 + 93 = 521$
ACBDEA	$78 + 112 + 88 + 96 + 103 = 477$
ACBEDA	$78 + 112 + 84 + 96 + 93 = 463$
ACEBDA	$78 + 110 + 84 + 88 + 93 = 453$
ACDBEA	$78 + 135 + 88 + 84 + 103 = 488$
ADBCEA	$93 + 88 + 112 + 110 + 103 = 506$
ADCBEA	$93 + 135 + 112 + 84 + 103 = 527$

The minimum cost Hamiltonian circuit is ACEBDA with a minimum distance travelled of 453 miles.

Counting (NP-Complete Problems) and The Need For Heuristic Algorithms

For a complete graph on 25 vertices (which is not a lot of vertices), there are $24!/2 = 310\,224\,200\,866\,619\,719\,680\,000$ Hamiltonian circuits.

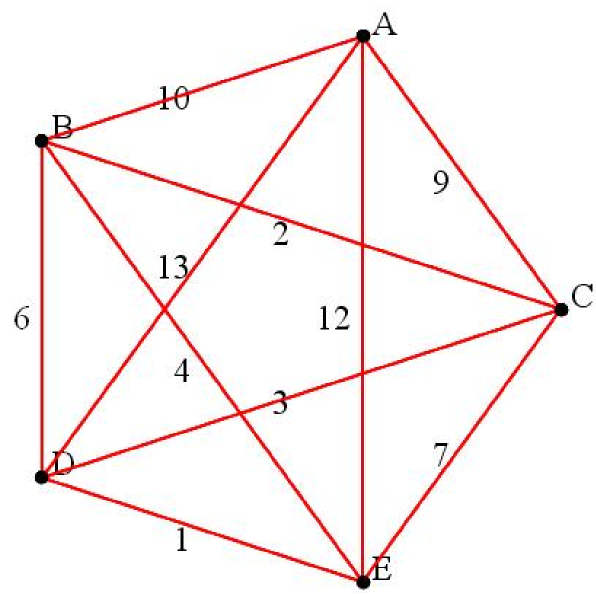
Consider a computer fast enough to calculate the length of one million Hamiltonian circuits per second.

Then it would take $\frac{1}{1\,000\,000}$ seconds to do one Hamiltonian circuit.

There are $60\frac{\text{seconds}}{\text{minute}} \times 60\frac{\text{minutes}}{\text{hour}} \times 24\frac{\text{hours}}{\text{day}} \times 365\frac{\text{days}}{\text{year}} = 31\,536\,000$ seconds per year.

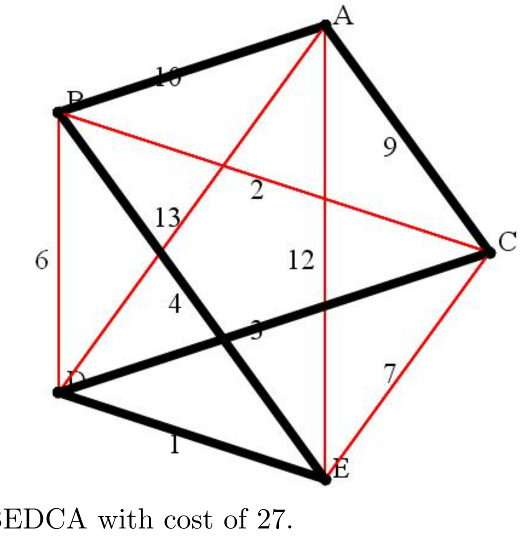
Therefore, the number of years it would take this computer to find the length of all the Hamiltonian circuits is $\frac{24!}{2}$ Hamiltonian circuits $\times \frac{1}{1\,000\,000}$ seconds $\times \frac{1}{31\,536\,000}\frac{\text{years}}{\text{second}} \sim 10\,000\,000\,000$ years, or ten billion years!

Example We are interested in finding the minimum cost Hamiltonian circuit for the following complete graph.



Since this is a complete graph with 5 vertices, it has $(5-1)!/2 = 12$ different Hamiltonian circuits.

Hamiltonian Circuit	Cost (Distance Travelled)
ABCDEA	28
ABCEDA	33
ABDCEA	38
ABDECA	33
ABEDCA	27
ABECDA	37
ACBDEA	30
ACBEDA	29
ACEBDA	39
ACDBEA	34
ADBCEA	40
ADCBEA	34



The minimum cost Hamiltonian circuit (shown above) is ABEDCA with cost of 27.