Tilings

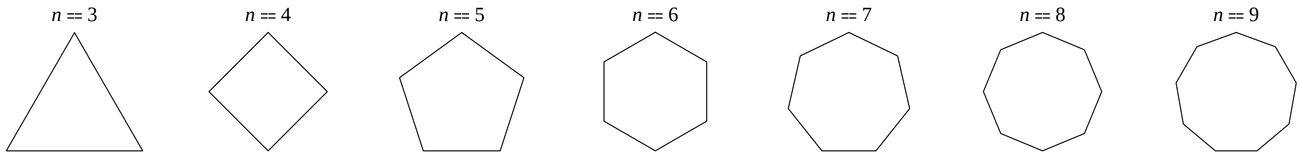
A tiling is a covering of the entire plane with figures which do not not overlap, and leave no gaps. Tilings are also called *tesselations*, so you will see that word often.

The plane is in two dimensions, and that is where we will focus our examination, but you can also tile in one dimension (over a line), in three dimensions (over a space) and mathematically, in even higher dimensions! How cool is that?

Monohedral tilings use only one size and shape of tile.

Regular Polygons

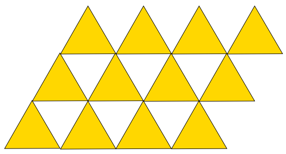
A regular polygon is a figure whose sides are all the same length and interior angles are all the same.



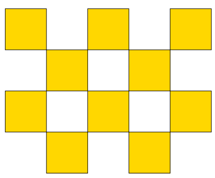
Equilateral Triangle, Square, Pentagon, Hexagon, n-gon for a regular polygon with n sides.

Not all of these regular polygons can tile the plane. The regular polygons which do tile the plane create a regular tiling, and if the edge of a tile coincides entirely with the edge of a bordering tile, it is called an edge-to-edge tiling.

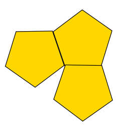
Triangles have an edge to edge tiling:



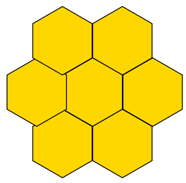
Squares have an edge to edge tiling:



Pentagons don't have an edge to edge tiling:



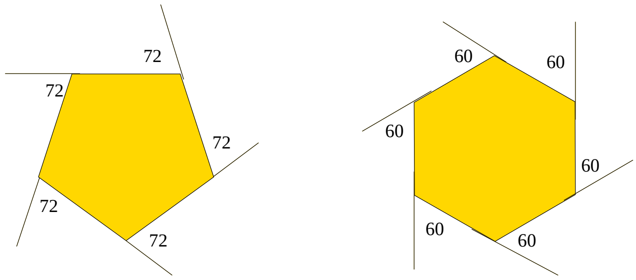
Hexagons have an edge to edge tiling:



The other n-gons do not have an edge to edge tiling.

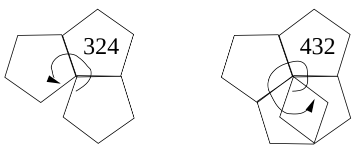
The problem with the ones that don't (pentagon and $n > 6$) has to do with the angles in the n-gon.

An n-gon will have exterior angles of $360/n$ (the sum of all the exterior angles must add up to 360 degrees). Consider the pentagon and hexagon:



Therefore, the interior angles of the hexagon are 120 degrees each. In an edge to edge tiling, at a point where the corners of hexagons meet, we would require the interior angles to sum to 360 degrees. This happens if we have 3 hexagons, and we see that our regular tiling indeed has three hexagons meeting at each corner.

The interior angles of the pentagon are 108 degrees each. At a point where the corners of pentagons meet, we would require the interior angles to sum to 360 degrees. Three pentagons would be 324 degrees, which is not enough, and four pentagons would be 432 degrees, which is too much. This is why we cannot tile the plane with pentagons.



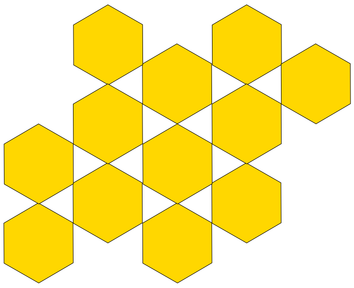
This also explains why we have four squares meeting at each corner for squares ($4 \times 90 = 360$), and six triangles meeting at each corner for triangles ($6 \times 60 = 360$).

The only edge to edge regular tilings are the ones with equilateral triangles, squares, or hexagons.

Semi-Regular Tiling

You can also combine tile shapes (say, for example, a square and a hexagon) and try to tile using those shapes. If every vertex (where shapes meet) uses the same set of regular polygons then it is called a semiregular tiling. For semiregular tilings, we still need the interior angles to add to 360 degrees at any point where the polygons meet.

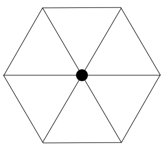
Example Here is a semiregular tiling that uses hexagons and equilateral triangles



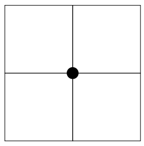
There are only a finite number (eight, in fact) semiregular tilings.

Naming Convention for Regular and Semiregular Tilings

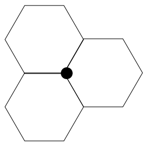
Tilings can be named by going around a vertex and listing the number of sides each regular polygon has. Here are the three regular and eight semiregular tilings and their names.



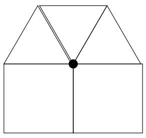
3,3,3,3,3,3



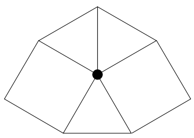
4,4,4,4



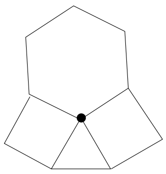
6,6,6



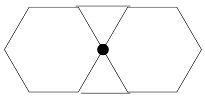
3,3,3,4,4



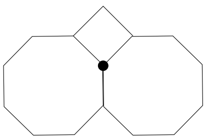
3,3,4,3,4



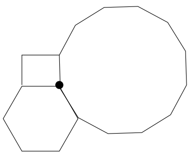
3,4,6,4



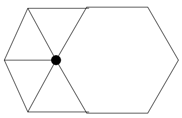
3,6,3,6



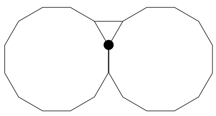
4,8,8



4,6,12

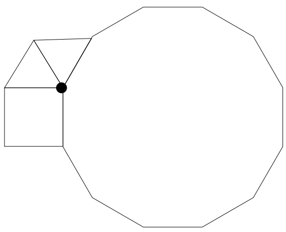


3,3,3,3,6



3,12,12

You can construct other combinations of regular polygons which fill the space around a point, but the shapes will not tile the plane. An example of this is the 12,4,3,3 combination shown here (try it!):

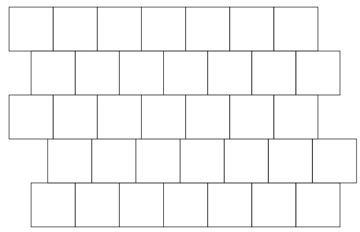


You can see the tiling that they produce here:
http://cda.morris.umn.edu/ mcquarrb/SurveyofMath/Resources/Tesselations.html.

If we do not require the vertex order to be the same at each vertex, we can construct infinitely many tilings.

Non Edge-to-Edge Tilings

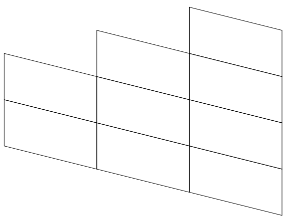
Example Here is an example of a tiling using squares which is not edge to edge:



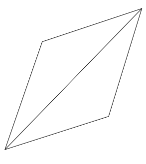
Irregular Polygon Tilings

A quadrilateral is a four sided polygon, and in general the sides are different lengths. Parallelograms are quadrilaterals whose opposite sides are parallel. Here is an example of a quadrilateral and a parallelogram.

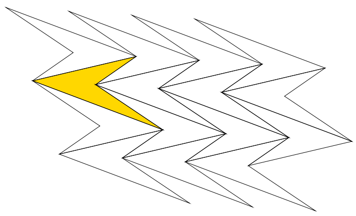
Parallelograms can tile the plane:



Triangles can always be be combined to make a parallelogram, so triangles will always be able to tile the plane.



What might be surprising is that any quadrilateral can tile the plane.



A <u>convex</u> shape means that for any two points in the shape or on the boundary of the shape, the line joining the two points lies within the shape or on the boundary of the shape.

For convex hexagons, there are only three classes of convex hexagons which tile the plane.

For convex pentagons, no one knows how many classes exist which tile the plane. There are at least fourteen classes, the latest of which was discovered in the 1980's.

The results are understood for n-gons with seven or more sides. A convex polygon with seven or more sides cannot tile the plane.