The mathematical concepts we use to describe finance are also used to describe how populations of organisms vary over time, how disease spreads through a population, how rumours spread through a population, even the motion of particles suspended in a fluid, as well as many other situations.

Mathematics is so beautiful because the techniques you learn to solve one type of problem typically can be used to solve other problems!

Money deposited in a savings account in a bank will earn interest. The initial amount you deposit is called the principal, and the money which is earned is called the interest.

How does the money grow? What will your balance be after one year? There is a lot that goes into answering these questions, since interest can be paid in different ways.

Growth of Savings: Simple Interest

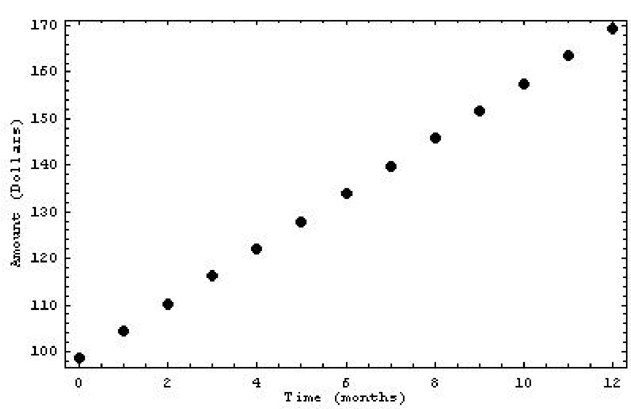
Simple interest pays interest only on the principal, not on any interest which has accumulated. Simple interest is rarely used for saving accounts, but it is used for bonds.

Example You put \$98.45 in a savings account which pays simple interest of 6% a month. How much money do you have in the savings account after 4 months?
Solution To answer this question, we can build from what we know. Simple interest means we pay interest only on the initial amount deposited (principal), which was \$98.45. The interest amount will be 6%= 6/100 = 0.06 of the principal, and added to the account balance once a month.

Interest Period	Date	Interest Added	Accumulated Amount
0	Jan 1	0	\$98.45
1	Feb 1	\$98.45×0.06=\$5.91	\$98.45 + \$5.91 = \$104.36
2	Mar 1	\$98.45×0.06=\$5.91	\$104.36 + \$5.91 = \$110.26
3	Apr 1	\$98.45×0.06=\$5.91	\$110.26 + \$5.91 = \$116.17
4	May 1	\$98.45×0.06=\$5.91	\$116.17 + \$5.91 = \$122.08

This table is the form an Excel spreadsheet would take to calculate simple interest. Notice the first row is an initialization and it is the second row that contains formulas.

We see that the growth is by a constant amount (\$98.45×0.06=\$5.91) every time period (month in this case). This is the requirement for linear or arithmetic growth. It gets the name linear since the graph of the amount versus the time is a straight line (linear function).



Simple Interest Formula

For simple interest of $r\%$ paid every time period with a principal P, we get

Time Period	Accumulated Amount
0	P
1	$(P) + Pr$
2	$(P + Pr) + Pr = P + 2Pr$
3	$(P + 2Pr) + Pr = P + 3Pr$
4	$(P + 3Pr) + Pr = P + 4Pr$
	$\vdots$
k	$P + Prk$

ie., for a principal of P with simple interest of $r\%$ paid every time period, we get an accumulated amount after k time periods of

$$A = P + Prk = P(1 + kr).$$

The formula gives you another way of calculating a quantity that could be done using a spreadsheet style table.

Example You put \$98.45 in a savings account which pays simple interest of 6% a month. How much money do you have in the savings account after 4 months?
Solution Identify $P = \$98.45$, $r = 6\%$ and time period of one month. So for four months, $k = 4$.

$$\begin{aligned} A &= P(1 + kr) \\ &= \$98.45(1 + 4 \times 0.06) \\ &= \$122.08 \end{aligned}$$

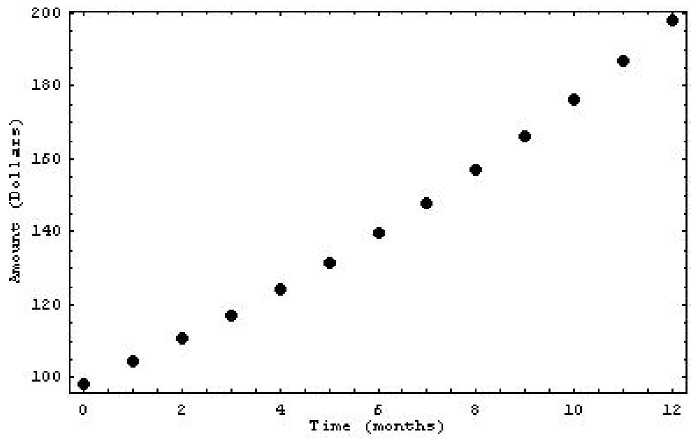
Growth of Savings: Compound Interest

Compound interest pays interest on the principal and the accumulated interest, not just the principal.

Example You put \$98.45 in a savings account which pays compound interest of 6% a month. How much money do you have in the savings account after 4 months?
Solution To answer this question, we can build from what we know. Compound interest means we pay interest on the accumulated amount in the account. The interest amount will be $6\% = 6/100 = 0.06$ of this amount, and added to the account balance once a month.

Compounding Period	Date	Interest Added	Accumulated Amount
0	Jan 1	0	\$98.45
1	Feb 1	\$98.45×0.06=\$5.91	\$98.45 + \$5.91 = \$104.36
2	Mar 1	\$104.36×0.06=\$6.26	\$104.36 + \$6.26 = \$110.62
3	Apr 1	\$110.62×0.06=\$6.64	\$110.62 + \$6.64 = \$117.26
4	May 1	\$117.26×0.06=\$7.04	\$117.26 + \$7.04 = \$124.29

We see that the amount of growth increases as time increases. The amount of growth is proportional to the amount present, which is the requirement for geometric growth.



Interest Terminology

Savings problems typically involve a bit more terminology than we've used so far.

The compounding period is the time which elapses before compound interest is paid.

The time when compounding is done effects the accumulated amount, since the current amount affects the amount of interest added, and the current amount will change if we compound more frequently.

The nominal rate is the stated rate of interest for a specified length of time. The nominal rate does not take into account how interest is compounded!

The effective rate is the actual percentage rate of increase for a length of time which takes into account compounding. It represents the amount of simple interest that would yield exactly as much interest over that length of time.

The effective annual rate (EAR) is the effective rate given over a year. For savings accounts, the EAR is also called the annual percentage yield (APY).

Compound Interest Formula

For a nominal annual rate r, compounded n times per year, we have $i = r/n$ as the interest rate per compounding period. Now let's try to derive a formula for compound interest.

Compounding Period	Amount
0	P
1	$P + Pi = P(1+i)$
2	$P(1+i) + P(1+i)i = P(1+i)^2$
3	$P(1+i)^2 + P(1+i)^2 i = P(1+i)^3$
4	$P(1+i)^3 + P(1+i)^3 i = P(1+i)^4$
	$\vdots$
k	$P(1+i)^k$

ie., for a principal of P with compound interest of $i\%$ paid every compounding period, we get an accumulated amount after k compounding periods of

$$A = P(1+i)^k.$$

Example You put \$98.45 in a savings account which pays compound interest of 6% a month. How much money do you have in the savings account after 4 months?
Solution Identify $P = \$98.45$, $i = 6\%$. So for four months, $k = 4$.

$$\begin{aligned} A &= P(1+i)^k \\ &= \$98.45(1+0.06)^4 \\ &= \$124.29 \end{aligned}$$

Example \$1000 is deposited at 6% per year. Find the balance at the end of one year, if the interest paid is a) simple interest b) compounded quarterly.
Solution
a) The principal is P =\$1000, and the nominal rate is r =6% = 0.06. After one year, $k = 1$. If we use simple interest, we have an accumulated balance of

$$A = P(1+kr) = \$1000.00(1+1\times 0.06) = \$1060.00$$

b) For interest compounded quarterly, we can get the accumulated amount by constructing a table. The $r = 6\%$, when compounded quarterly, means we get $i = r/n = 6\%/4 = 0.015$ interest applied every three months over the year.

Compounding Period	Date	Interest Added	Accumulated Amount
0	Jan 1, 2005	0	\$1000.00
1	Apr 1, 2005	\$1000.00×0.015=\$15.00	\$1000.00 + \$15.00 = \$1015.00
2	Jul 1, 2005	\$1015.00×0.015=\$15.23	\$1015.00 + \$15.23 = \$1030.23
3	Oct 1, 2005	\$1030.23×0.015=\$15.45	\$1030.23 + \$15.45 = \$1045.68
4	Jan 1, 2006	\$1045.68×0.015=\$15.68	\$1045.68 + \$15.68 = \$1061.36

Since after one year, the interest earned was \$61.36, which is 6.136% of the principal, the APR is 6.136%.
We could also use a formula to answer part (b).
b) The nominal annual rate is r =6% = 0.06, when compounded quarterly, means we have $n = 4$, so $i = r/n = 0.06/4 = 0.015$. One year corresponds to $k = 4$, so after one year we have

$$A = P(1+i)^k = \$1000.00(1+0.015)^4 = \$1061.36.$$

The formulas allow us to answer questions which would be difficult to answer using a table.

Example \$1000 is deposited at 7.5% per year. Find the balance at the end of one year, and two years, if the interest paid is compounded daily.
Solution
The nominal annual rate is r =7.5% = 0.075, when compounded daily, means we have $n = 365$, so $i = r/n = 0.075/365 = 0.000205479$.
One year corresponds to $k = 365$, so after one year we have

$$A = P(1+i)^k = \$1000.00(1+0.000205479)^{365} = \$1077.88.$$

Two years corresponds to $k = 2\times 365 = 730$, so after two years we have

$$A = P(1+i)^k = \$1000.00(1+0.000205479)^{730} = \$1161.82.$$

A Limit to Compounding

(This is Table 21.1 in the text) Sketch the graph of the accumulated amount for 10 years if the principal is $P =$\$1000 and the annual interest rate is $r = 10\%$ for simple interest, compound interest compounded yearly, compound interest compounded quarterly, and compound interest compounded daily (assume 365 days in a year).

To get the values, we can use the formulas we derived. Here is the process for getting a point beyond $t = 0$; the rest are calculated in a similar fashion.

Simple interest after 1 year:

$$A = P(1 + rt) = \$1000(1 + 0.10 \times 1) = \$1100.00 \text{ after 1 year.}$$

Compound interest compounded yearly ($i = r/n = 0.10/1 = 0.10$, and $k = 1$):

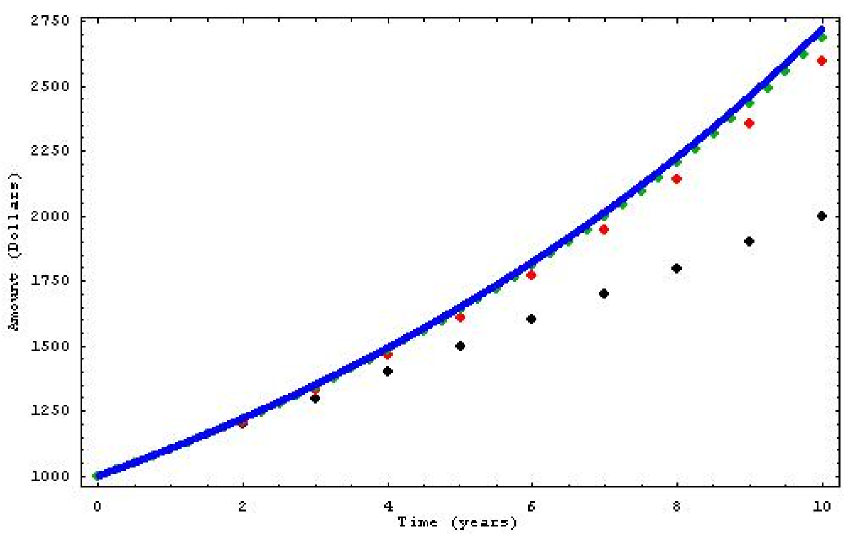
$$A = P(1 + i)^k = \$1000(1 + 0.10)^1 = \$1100.00 \text{ after 1 year.}$$

Compound interest compounded quarterly ($i = r/n = 0.10/4 = 0.025$, and $k = 4$):

$$A = P(1 + i)^k = \$1000(1 + 0.025)^4 = \$1103.81 \text{ after 1 year.}$$

Compound interest compounded daily ($i = r/n = 0.10/365 = 0.000273973$, and $k = 365$):

$$A = P(1 + i)^k = \$1000(1 + 0.000273973)^{365} = \$1105.16 \text{ after 1 year.}$$



black: simple interest.
red: compound interest, compounded yearly.
green: compound interest, compounded quarterly.
blue: compound interest, compounded daily.

Things to note:

- The curves are all essentially the same for short times.
- There are more points for compounding quarterly than yearly since interest is paid more often during the year.
- There is not much difference over 10 years to compounding quarterly and compounding daily.

Compounding more frequently leads to a larger accumulated balance, but there is a limit to this process. The limit would be if we compounded continuously.

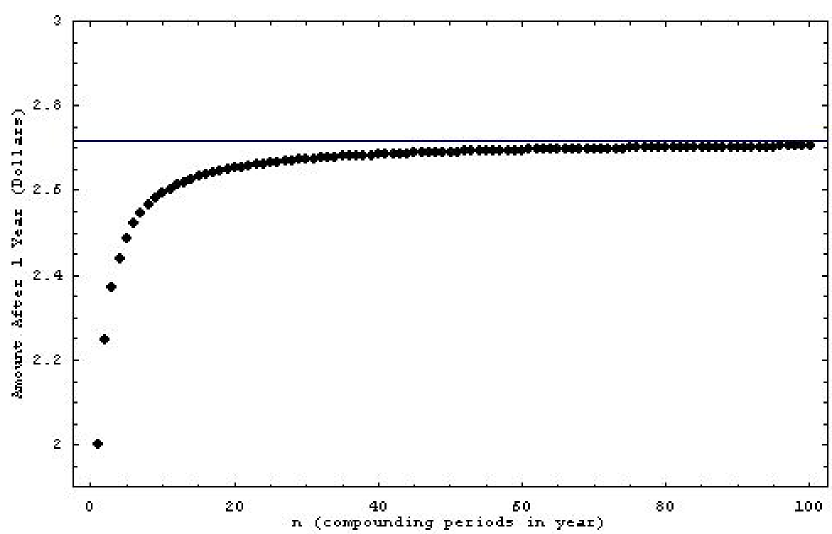
Compounding Continuously

Consider a principal $P = \$1$ and a rate of r=100% which is compounded over shorter and shorter time periods. We are interested in how much the accumulated amount will be after one year.

Compound interest compounded n times a year ($i = 1/n$, and $k = n$ (to get one year)):

$$A = P(1+i)^k = \left(1 + \frac{1}{n}\right)^n \text{ after 1 year.}$$

There is a table in the text of these numbers, here is a sketch



We see that the accumulated amount is approaching a number:

$$\left(1 + \frac{1}{n}\right)^n \sim 2.71828\ldots \text{ if } n \text{ is very large.}$$

This number is similar to $\pi = 3.14...$ in that it is mathematically significant and appears in many situations, and so we give it a special designation:

$$e \sim \left(1 + \frac{1}{n}\right)^n \sim 2.71828\ldots \text{ if } n \text{ is very large.}$$

This leads the the continuous interest formula, which is

$A = Pe^r$ after 1 year if interest is compounded continuously at annual rate r.

$A = Pe^{rt}$ after t years if interest is compounded continuously at annual rate r.

The function e^{rt} is called the exponential function.

The continuous interest formula is the upper limit on the accumulated amount that can accrue due to compounding interest.

Review of interest formulas (principal P and annual rate r)

- Simple interest: $A = P(1 + kr)$ is the amount at time k.
- Compound interest, compounded n times over 1 year: $A = P(1 + i)^k$ is the amount at time k, where $i = r/n$.
- Continuously compounded interest: $A = Pe^{rt}$ is the amount at time t.

The formulas allow us to answer questions which would be difficult to answer using a table, and also to answer questions quickly without a lot of calculation. However, the tables allow us to answer questions that do not match the conditions under which the formulas were derived. Therefore, both formulas and spreadsheet tables are useful in understanding how personal finance works.