A Limit to Compounding

Example (This is Table 21.1 in the text) Sketch the graph of the accumulated amount for 10 years if the principal is $P =$\$1000 and the annual interest rate is $r = 10\%$ for simple interest, compound interest compounded yearly, compound interest compounded quarterly, and compound interest compounded daily (assume 365 days in a year).

To get the values, we can use the formulas we derived. Here is the process for getting a point beyond $t = 0$; the rest are calculated in a similar fashion.

Simple interest after 1 year:

$$A = P(1 + rt) = \$1000(1 + 0.10 \times 1) = \$1100.00 \text{ after 1 year.}$$

Compound interest compounded yearly ($i = r/n = 0.10/1 = 0.10$, and $k = 1$):

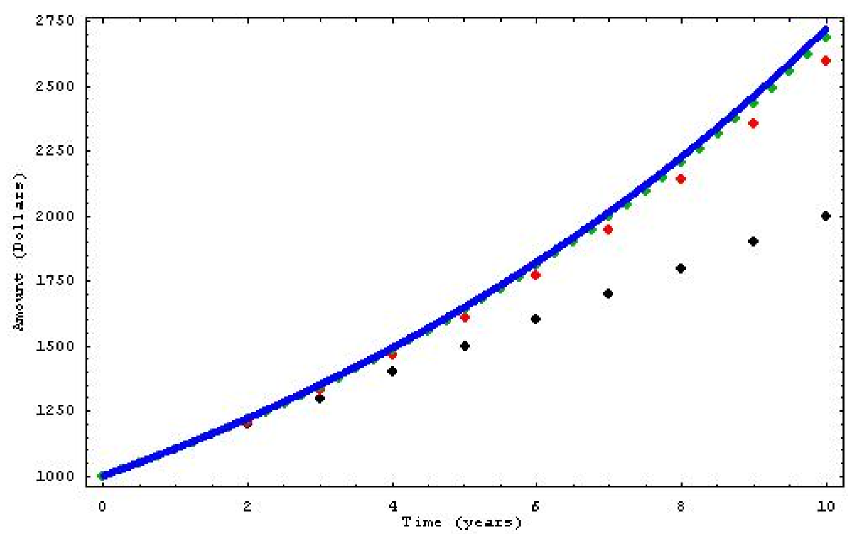
$$A = P(1 + i)^k = \$1000(1 + 0.10)^1 = \$1100.00 \text{ after 1 year.}$$

Compound interest compounded quarterly ($i = r/n = 0.10/4 = 0.025$, and $k = 4$):

$$A = P(1 + i)^k = \$1000(1 + 0.025)^4 = \$1103.81 \text{ after 1 year.}$$

Compound interest compounded daily ($i = r/n = 0.10/365 = 0.000273973$, and $k = 365$):

$$A = P(1 + i)^k = \$1000(1 + 0.000273973)^{365} = \$1105.16 \text{ after 1 year.}$$



black: simple interest.
red: compound interest, compounded yearly.
green: compound interest, compounded quarterly.
blue: compound interest, compounded daily.

Things to note:

- The curves are all essentially the same for short times.
- There are more points for compounding quarterly than yearly since interest is paid more often during the year.
- There is not much difference over 10 years to compounding quarterly and compounding daily.

Compounding more frequently leads to a larger accumulated balance, but there is a limit to this process. The limit would be if we compounded continuously.

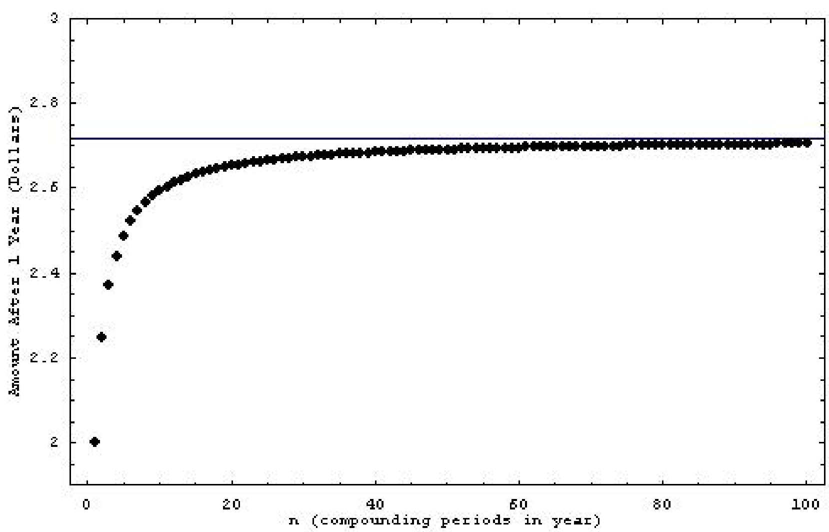
Compounding Continuously

Example Consider a principal $P = \$1$ and a rate of r=100% which is compounded over shorter and shorter time periods. We are interested in how much the accumulated amount will be after one year.

Compound interest compounded n times a year ($i = 1/n$, and $k = n$ (to get one year)):

$$A = P(1+i)^k = \left(1 + \frac{1}{n}\right)^n \text{ after 1 year.}$$

There is a table in the text of these numbers, here is a sketch



We see that the accumulated amount is approaching a number:

$$\left(1 + \frac{1}{n}\right)^n \sim 2.71828... \text{ if } n \text{ is very large.}$$

This number is similar to $\pi = 3.14...$ in that it is mathematically significant and appears in many situations, and so we give it a special designation:

$$e \sim \left(1 + \frac{1}{n}\right)^n \sim 2.71828... \text{ if } n \text{ is very large.}$$

This leads the the continuous interest formula, which is

$A = Pe^r$ after 1 year if interest is compounded continuously at annual rate r.

$A = Pe^{rt}$ after t years if interest is compounded continuously at annual rate r.

The function e^{rt} is called the exponential function.

The continuous interest formula is the upper limit on the accumulated amount that can accrue due to compounding interest.

Review of interest formulas (principal P and annual rate r):

- Simple interest: $A = P(1 + rt)$ is the amount at time t.
- Compound interest, compounded n times over 1 year: $A = P(1 + i)^k$ is the amount at time k, where $i = r/n$.
- Continuously compounded interest: $A = Pe^{rt}$ is the amount at time t.