Geometric Series

Consider the following series: $1 + x + x^2 + x^3 + \cdots + x^{n-1}$.

We need to figure out a way to write this without the $\cdots$. Here's how:

$$\begin{aligned}
s &= 1 + x + x^2 + x^3 + \cdots + x^{n-1} \\
\text{subtract } \underline{xs} &= \underline{\phantom{1+{}} x + x^2 + x^3 + \cdots + x^{n-1} + x^n} \\
s - sx &= 1 - 0 - 0 - 0 - \cdots - 0 - x^n \\
s - sx &= 1 - x^n
\end{aligned}$$

Now solve for s:

$$s(1-x) = 1 - x^n \Rightarrow s = \frac{1-x^n}{1-x} = \frac{x^n - 1}{x-1}.$$

Therefore, a <u>geometric series</u> has the following sum: $1 + x + x^2 + x^3 + \cdots + x^{n-1} = \dfrac{x^n - 1}{x-1}$.

Exponential and Natural Logarithms

As we have seen, some of our equations involve exponents. To effectively deal with exponents, we need to be able to work with exponentials and logarithms.

The exponential and natural logarithm functions are inverse functions, and related by the following rules

$$\begin{aligned}
e &= \left(1 + \frac{1}{n}\right)^n \text{ for } n \text{ large} \\
e &= 2.71828\ldots \\
\ln(e^A) &= A \text{ where } A \text{ is a constant} \\
e^{\ln(A)} &= A
\end{aligned}$$

If the base of the exponent is not e, we can use the following rule:

$$\begin{aligned}
\ln(b^A) &= A\ln(b) \text{ where } A \text{ and } b \text{ are constants} \\
\ln(2^A) &= A\ln(2) \\
\ln((1+r)^A) &= A\ln(1+r)
\end{aligned}$$

Note:

- The natural logarithm acts on a number, so we read $\ln(2)$ as "The natural logarithm of 2".
- We would never write $\ln(2) = 2\ln$, since this loses the fact that the natural logarithm must act on something. It is a functional evaluation, not a multiplication.
- This is similar to trigonometric functions, like $\sin(\pi)$. The sine function is being evaluated at π when we write $\sin(\pi)$, just as the natural logarithmic function is being evaluated at 2 when we write $\ln(2)$.

Example Solve the equation $(1.002)^k = 1.0832$ for k.
Solution We will need to use the natural logarithm here:

$$\begin{aligned} 1.002^k &= 1.0832 \\ \ln(1.002^k) &= \ln(1.0832) \\ k\ln(1.002) &= \ln(1.0832) \\ k &= \frac{\ln(1.0832)}{\ln(1.002)} = 39.9998 \sim 40 \end{aligned}$$

Accumulation

An important aspect of saving is the idea of accumulation, which answers the question: *What size deposit do I have to make at regular time interval d to save a certain amount of money in a certain amount of time?* This would be important for saving for retirement, or a down payment on a house, or a car, or a child's education.

Obviously, if there was no interest, you would just break the amount you need to save into d even pieces and deposit that amount regularly. Interest makes the problem more interesting!

Example You begin saving for retirement at age 35 by paying \$100 a month into an account paying 6% annual interest compounded monthly. How much will you have in savings by the time you are 65?

The easiest way to think of this is backwards, starting by what happens at age 65. For interest compounded monthly at an annual rate of 6%, we have $i = r/n = 0.06/12 = 0.005$.

The last deposit you make will be \$100, and earn no interest (or interest for 0 months). \$100

The penultimate deposit will be \$100, and will earn interest for 1 month: $\$100(1+i)^1$.

The second last deposit will be \$100, and will earn interest for 2 month: $\$100(1+i)^2$.

This process continues, right up until the first deposit is made. In $65 - 35 = 30$ years, you will make $30 \times 12 = 360$ monthly deposits.

The amount you save is

$$A = \$100 + \$100(1+i)^1 + \$100(1+i)^2 + \cdots + \$100(1+i)^{359} = \$100\left[1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{359}\right].$$

We stop at 359 since we started at 0, not 1.

This is a geometric series, with $x = (1+i)$ and $n = 360$.

Therefore, we can write

$$A = \$100\left[1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{359}\right] = \$100\left[\frac{(1+i)^{360} - 1}{(1+i) - 1}\right] = \$100\left[\frac{(1+i)^{360} - 1}{i}\right].$$

The amount we will save by the age of 65 is

$$A = \$100\left[\frac{(1+i)^{360}-1}{i}\right] = \$100\left[\frac{(1+0.005)^{360}-1}{0.005}\right] = \$100\,451.50.$$

Only \$36, 000 of this is due to the deposits. The rest is interest.

We now have a new formula: for a uniform deposit d per compounding period and an interest rate of i per period, the amount A accumulated after k periods is given by the savings formula: $\boxed{A = d\left[\frac{(1+i)^k-1}{i}\right]}$

Example What should your monthly deposit be in a savings account with 7% annual interest compounded monthly if you want to save \$3000 in 10 months for the down payment on a new car?
Solution We use the savings formula, since we know $A = \$3000$, $i = 0.07/12 = 0.00583333$, and $k = 10$ is the time period. We need to figure out d.

$$\begin{aligned}
A &= d\left[\frac{(1+i)^k-1}{i}\right]\\
\$3000 &= d\left[\frac{(1+0.00583333)^{10}-1}{0.00583333}\right]\\
\$3000 &= d\,[10.2666]\\
\frac{\$3000}{10.2666} &= d\\
\$292.21 &= d
\end{aligned}$$

So you will have to save \$292.21 per month to have a \$3000 down payment in 10 months.

Example You wish to remodel your kitchen, and estimate it will cost \$35,000 to do. If you can afford to save \$500 a month in a savings account that earns 4% annual interest, how long will it take you to save enough to remodel the kitchen?
Solution We use the savings formula, with $A = \$35,000$, $i = 0.04/12 = 0.00333333$, and $d = \$500$. We need to figure out k, the number of months it will take to save \$35,000.

$$\begin{aligned}
A &= d\left[\frac{(1+i)^k-1}{i}\right]\\
\$35000 &= \$500\left[\frac{(1+0.00333333)^k-1}{0.00333333}\right]\\
\frac{\$35000\times 0.00333333}{\$500} &= 1.00333333^k-1 && \text{first, isolate the } 1.003333^k\\
0.233333 &= 1.00333333^k-1\\
1+0.233333 &= 1.00333333^k\\
1.233333 &= 1.00333333^k && \text{now we take the natural logarithm}\\
\ln(1.233333) &= \ln(1.00333333^k)\\
\ln(1.233333) &= k\ln(1.00333333) && \text{using our logarithm rule } \ln(b^A) = A\ln(b)\\
k &= \frac{\ln(1.233333)}{\ln(1.00333333)} && \text{solve for } k\\
k &= 63.027
\end{aligned}$$

So it will take 63 months (5 years 3 months) to save for the kitchen remodel.

Exponential Decay

Exponential Decay is geometric growth with a negative rate of growth.

If $i > 0$, then

growth: $A = P(1+i)^k$ $\quad A$ is the accumulated amount

decay: $V = P(1-i)^k$ $\quad V$ is the value

This decrease in the amount models inflation over a short time period, where the value of the dollar goes down geometrically, or depreciation, where the value of an item decreases.

You can think of the value of a dollar as depreciating over time much like the value of an item depreciates over time (cars are a prime example of an item whose value decreases over time).

The actual price of an item at any time is said to be in current dollars. To compare prices of items from different times (which will take into account inflation), we use constant dollars, which are dollars from a particular year.

Example Suppose you bought a car in early 2002 for \$10,000. If its value (in current dollars) depreciates steadily at 12% per year (cars typically depreciate at 15-20% a year), what will be its value (in current dollars) in early 2005?
Solution The car was bought in 2002 and we want to know something about the value of the car in 2005, which is 3 years later.
The value of the car depreciates at 12% per year, so $i = 0.12/1 = 0.12$. Since the value of the car is decreasing, we use the formula for V (value):

$$\begin{aligned} V &= P(1-i)^k \\ &= \$10000(1-0.12)^3 \\ &= \$10000(0.88)^3 \\ &= \$10000(0.88)(0.88)(0.88) \\ &= \$6814.72 \end{aligned}$$

Notice that rounding numbers early can significantly change your answer!!!
If you say $(0.88)^3 = 0.68$ instead of $(0.88)^3 = 0.681472$ you get a final value of \$6800.00, which is wrong due to rounding errors.

The Consumer Price Index

If inflation stayed constant over the years, we could use the above ideas to compare the cost of an item in an earlier year with the cost of the item today. However, inflation is not a constant! It varies over time.

The Consumer Price Index (CPI) (http://www.bls.gov/cpi/home.htm) allows us to compare the cost of items in different years.

The CPI represents costs of a basket of goods (food, housing, transportation, etc). This cost is measured each year for the same set of goods. The cost will vary over time, and also over region.

There must be some base number against which all the other numbers are compared, so the CPI for the years 1982–1984 is set to 100 (this is arbitrarily chosen by the Bureau of Labor, they could have chosen something else).

If you want to relate the cost of two items in different years you use the relation:

$$\frac{\text{Cost in Year } A}{\text{Cost in Year } B} = \frac{\text{CPI in Year } A}{\text{CPI in Year } B}$$

There are different CPI for different regions or metro areas, and also for different sets of goods.

The FAQ on the CPI website contains a wealth of information about how to use the CPI effectively.

Example What is the value of a dollar from 1970 in 1987 dollars?
Solution

$$\frac{\text{Cost in 1970}}{\text{Cost in 1987}} = \frac{\text{CPI in 1970}}{\text{CPI in 1987}}$$

$$\frac{\text{Value of \$1 in 1970}}{\text{Value of \$1 in 1987}} = \frac{38.8}{113.6}$$

$$\text{Value of \$1 in 1970} = 0.341549\text{Value of \$1 in 1987}$$

$$\text{Value of \$1 in 1970} = \text{ \$0.341549 in 1987}$$

So the value of a 1970 dollar would have been 34 cents in 1987. Another way of saying this is that $1 in 1970 has the same buying power as $0.34 in 1987.

Example When buying a new home, Sam learns from her parents that they paid $39,000 in 1967 for the house she grew up in. Seeing that houses cost much more today, Sam tells her parents that they got an incredibly good deal on their house, and she is spending much more on her $150,000 house today.
Her parents chuckle, and tell Sam that the cost of items has gone up over the years, and if she really wants to compare the cost of her childhood home to her new house, she needs to take that into account.
How much would Sam's childhood home be in 2004 dollars?
Solution

$$\frac{\text{Cost of house in 2004}}{\text{Cost of house in 1967}} = \frac{\text{CPI in 2004}}{\text{CPI in 1967}}$$

$$\frac{\text{Cost of house in 2004}}{\text{Cost of house in 1967}} = \frac{188.9}{33.4} = 5.5.65569$$

$$\text{Cost of house in 2004} = 5.46707 \times \text{Cost of house in 1967} = 5.65569 \times \$39000 = \$220,572.$$