APR and EAR

Borrowing is saving looked at from a different perspective. The idea of simple interest and compound interest still apply.

A new term is the annual percentage rate (APR), which is the rate of interest per compounding period times the number of compounding periods per year.

Recall: i is the rate of interest per compounding period. We have calculated $i = r/n$, where r is the annual rate of interest, and n is the number of compounding periods per year. Therefore,

$$\text{APR } = i \times n = r.$$

Note that in general r might not be an annual rate of interest, which is why we have the APR terminology.

The APR is not the effective annual rate (EAR), as the following shows. Recall the EAR is the amount of simple interest that would produce the same accumulated amount at the end of the year.

Example Calculate the APR and EAR for a credit card on which interest per month is charged at 1.4%. Ignore late payment charges and other fees.
Solution The APR is $i \times n = 1.4\% \times 12 = 16.8\%$.
For the effective annual rate (EAR), we need to do some work.
The compound interest formula is $A = P(1+i)^k$. We will assume there are no charges which will accrue if we do not make any payments on the card. $k = 12$ since 12 compounding periods (months) equal one year. Note we don't need to know the principal P to get the EAR!

$$\begin{aligned} A &= P(1+i)^k \quad \text{compound interest formula} \\ &= P(1+0.014)^{12} \\ &= P(1.18156) \\ &= 1.18156\, P \end{aligned}$$

We want to know the simple interest rate that would lead to this accumulated amount. We use the simple interest formula with $k = 1$ year to determine that.

$$\begin{aligned} A &= P(1+kr) \quad \text{simple interest formula} \\ 1.18156\, P &= P(1+1\times r) \quad \text{solve for } r \\ 1.18156 &= 1+r \\ 1.18156-1 &= r \\ 0.18156 &= r \end{aligned}$$

Therefore, the EAR is 0.18156.
Another way to calculate the EAR is to figure out the interest earned as a percentage of the original principal: The actual interest paid is $1.18156\, P - P = 0.18156\, P$. This is $\dfrac{0.18156\, P}{P} = 0.18156 = 18.156\%$ of the principal.

The APR underestimates the true cost of borrowing, since it does not take into account compounding. The EAR is the true cost of borrowing.

Credit card companies are required to report the APR, which is fine since as long as each company reports the same thing consumers can compare companies on a level playing field, and quickly understand the differences between each account.

Conventional Loans

In a conventional loan, each payment pays towards the current interest that would be due over the life of the loan and also repays part of the principal. The payments are expressed in terms of an amortization table, which shows how much of each payment is going towards interest and how much towards paying off the principal.

Example You borrow \$100,000 at 8% per year for a 30 year loan for a house which will be paid off in (equal) monthly instalments. How much is your monthly payment?

Let's check this out initially using one of the many online resources: http://ray.met.fsu.edu/∼bret/amortize.html

We should find that $d = \$733.764$.

To figure out how the table is calculated, we use both the compound interest formula and the savings formula.

Thought One: This situation can be thought of as borrowing the entire \$100,000 immediately, and then putting it in a savings account where it will earn interest for 30 years until it (the interest and principal) has to be repaid.

$$\begin{aligned} A &= P(1+i)^k \\ &= \$100,000(1+0.08/12)^{360} \\ &= \$1,093,572.97 \end{aligned}$$

The amount you will have saved (which is the amount that you will have to repay) in 30 years will be \$1,093,572.97.

Thought Two: Saving $\$d$ each month (that's what we want to find!) for 30 years means that you will save:

$$\begin{aligned} A &= d\left[\frac{(1+i)^k - 1}{i}\right] \\ &= d\left[\frac{(1+0.08/12)^{360} - 1}{0.08/12}\right] \\ &= d\left[\frac{9.93573}{0.00666667}\right] \\ &= d(1490.36) \end{aligned}$$

We want these two amounts to be exactly equal.

$$\$1,093,572.97 = 1490.36d \longrightarrow d = \$733.764$$

The monthly payments should be \$733.76.

What we have done is called amortize the loan. Part of each monthly payment goes towards reducing the principal, and part goes toward reducing the interest that the loan would accumulate over the life of the loan.

Note that the million dollars itself is not what is being paid for the home, since the home is being paid off over the course of time and the principal is being reduced as time goes on.

The cost of the house is $\$733.764 \times 360 = \$264,153.60$, where \$164,153.60 is due to interest. There is a small correction made at the end due to the rounding that has been done.

The Amortization Formula

Leaving things in general in the example above, we see that:

$$P(1+i)^k = d\left[\frac{(1+i)^k - 1}{i}\right]$$

where P is the principal, i is the interest rate per compounding period, and k is the number of compounding periods for which you are taking out the loan.

A little algebra can be used to rewrite this as the Amortization Formula:

$$P = d\left[\frac{1-(1+i)^{-k}}{i}\right]$$

The amortization formula is used to determine the monthly payments d on a conventional loan.

Constructing The Amortization Table

First payment: \$733.76:

Interest for first month on the principal is $P \times i = P \times r/n = \$100,000 \times 0.08/12 = \$666.67$.

What is left goes towards reducing the principal: $\$733.76 - \$666.67 = \$67.09$.

At the end of the first month, the principal is $\$100,000 - \$67.09 = \$99932.91$.

Second Payment: \$733.76:

Interest for second month on the principal is $P \times i = P \times r/n = \$99932.91 \times 0.08/12 = \$666.22$.

What is left goes towards reducing the principal: $\$733.76 - \$666.22 = \$67.54$.

At the end of the first month, the principal is $\$99932.91 - \$67.54 = \$99865.37$.

This continues until the loan is paid off. The amount you are paying per month towards principal increases, and the amount you are paying towards interest decreases.

Note: If you look at FAQ #7 on the website http://ray.met.fsu.edu/~bret/amortize.html you will see the need for creating your own spreadsheet for the amortization table.

Verifying the ditech.com Ad The ad we saw in class said the monthly payments would increase by 11% if the APR changed by 1%. Let's verify this.

Use the amortization formula: $P = d\left[\frac{1-(1+i)^{-k}}{i}\right]$

$P = \$200,000$
APR $= 6\% = r$

$k = 360$ (360 months = 30 years)
$i = r/12 = 0.06/12 = 0.005$
d is the monthly payment

$$\begin{aligned}
\$200,000 &= d\left[\frac{1-(1.005)^{-360}}{0.005}\right] \\
\$200,000 &= d\,[166.792] \\
d &= \$1199.10
\end{aligned}$$

Redo the calculation, with APR = r =7%:
$i = r/12 = 0.07/12 = 0.00583333$

$$\begin{aligned}
\$200,000 &= d\left[\frac{1-(1.00583333)^{-360}}{0.00583333}\right] \\
\$200,000 &= d\,[150.308] \\
d &= \$1330.60
\end{aligned}$$

So the percentage increase is given by $\dfrac{\$1330.60 - \$1199.10}{\$1199.10} = 0.109666 \sim 11\%$.

Home Equity

The amount needed for a mortgage payment are generally larger than the amount needed to amortize a loan since the buyer must also pay taxes and insurance on their monthly mortgage. Read the example of page 843 for an example of a median priced home for a family with median income.

For what follows, we will ignore these details, as well as any increase in value of the house. The downpayment simply reduces the size of the loan that is required, so we can ignore it for now. The downpayment will simply add to the equity you have in your house.

We know that when a home loan is amortized, you pay towards the interest and the principal with each payment. Your early payments are heavily weighted towards the interest, and very little goes towards reducing the principal. Equity refers to the amount of money you have paid towards paying off the principal in your home.

If you have used a spreadsheet to construct an amortization table, then you already have the equity–it is the cumulative principal. Alternately, you can calculate equity at any given time using formulas in the following manner.

Example You purchase a home for $\$89,000$ with an annual interest rate of 6.375% and a 30 year mortgage. How much equity do you have in the house after 5 years?

First, we need to know how much the monthly payment is for this house. We can use the amortization formula to figure this out:

$$\begin{aligned}
P &= d\left[\frac{1-(1+i)^{-k}}{i}\right] \\
\$89,000 &= d\left[\frac{1-(1+0.06375/12)^{-360}}{(0.06375/12)}\right]
\end{aligned}$$

$$\begin{aligned}\$89,000 &= d\,[160.29] \\ d &= \frac{\$89,000}{160.29} = \$555.244\end{aligned}$$

Now, for the equity. The principal at time 0 is the entire $89,000, and the principal after 360 payments should be $0:

$$\begin{aligned}P &= d\left[\frac{1-(1+i)^{-(360-0)}}{i}\right] \\ &= \$555.244\left[\frac{1-(1+0.06375/12)^{-(360)}}{(0.06375/12)}\right] \\ &= \$89,000\end{aligned}$$

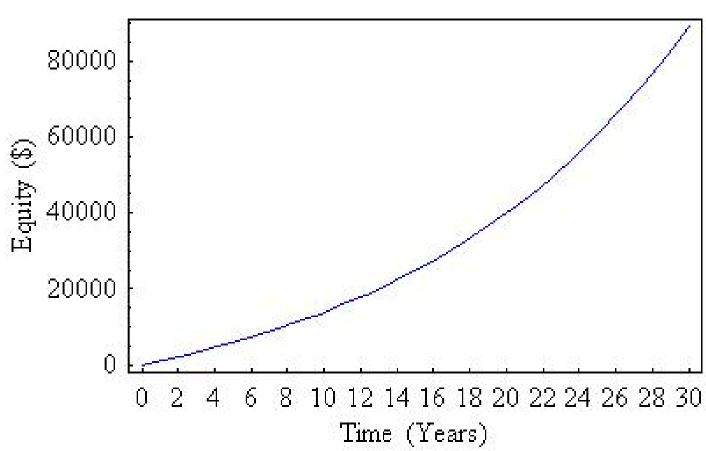
$$\begin{aligned}P &= d\left[\frac{1-(1+i)^{-(360-360)}}{i}\right] \\ &= \$555.244\left[\frac{1-1}{(0.06375/12)}\right] \\ &= \$0\end{aligned}$$

The above equations, which only change in the exponent, motivates the following. Now we can use the amortization formula again, this time to figure out what the principal is after 5 years or $5 \times 12 = 60$ months.

$$\begin{aligned}P &= d\left[\frac{1-(1+i)^{-(360-60)}}{i}\right] \\ &= \$555.244\left[\frac{1-(1+0.06375/12)^{-(360-60)}}{(0.06375/12)}\right] \\ &= \$83,192.34\end{aligned}$$

Yikes! We have only paid $\$89,000.00 - \$83,192.34 = \$5807.66$ towards the principal, which is the equity we have built up after 5 years.

The equity builds slowly initially, and then grows faster near the end of the mortgage period. Here is a plot of how equity varies over time for this example:



Interest Only Loans

The ad we saw for Quicken Loans appears to be an interest only loan. In an interest only loan, you reduce the monthly payments by not paying anything towards principal (hence the name). Of course, this reduces your monthly payment, but at the price of building equity. Until you start paying towards the principal, you are not building any equity in your home through payments towards principal–any equity you are building is based on the fact that home prices are increasing at a rate greater than inflation. That may not always be the case, of course.

After an initial time period (could be 10 years, but it will vary) you will have substantially higher monthly payments since you need to start repaying principal, and the time for the repayment is decreased from 30 year to 20 years.

Interest only loans are not a good idea for people whose income is unlikely to change over time. They are only a good idea if you anticipate a significant jump in your income that will allow you to pay the higher payments that are required later, or if you plan on flipping the house quickly before the higher payments kick in.

Let's say we have a loan structured in the following way. $P = \$100{,}000$ with an APR of $r = 8\%$ for 30 years, the first 10 of which are interest only.

For the first 10 years, you only pay interest, $P \times i = P \times r/n = \$100,000 \times 0.08/12 = \$666.67$. During this time, you have paid nothing towards the principal, so after 10 years you need an amortization loan for 20 years for the full purchase price:

$k = 12 * 20 = 240$ months
$i = r/n = 0.08/12 = 0.006666$

$$\begin{aligned} P &= d\left[\frac{1-(1+i)^{-k}}{i}\right] \\ \$100,000 &= d\left[\frac{1-(1.0066666)^{-240}}{(0.0066666)}\right] \\ \$100,000 &= d\,[119.551] \\ d &= \frac{\$100,000}{119.551} = \$836.47 \end{aligned}$$

The upshot of all this is that after 30 years, you pay more interest for the interest only loan than if you had used a traditional 30 year amortization. So in the long run, an interest only loan only makes sense if your income is going increase dramatically and you can make substantial payments to the principal later on (which will reduce the amount of interest you ultimately pay).

If we try to verify the numbers in the Quicken Loans ad, we will find they are wrong. The monthly payment for a traditional 30 year amortization of $P = \$150,000$ at an APR of 7.5% is $d = \$1048.42$ (they got it right on their website, although they did choose to round to \$1049 instead of \$1048). The interest only payment I calculate for this loan is $P \times r/n = \$150,000 \times 0.075/12 = \$937.50$, which does not agree with the \$745 in the TV ad or \$703 on their website. We would need to look a little more closely at how this loan is structured to fully understand it.