The result of many thousands of chance outcomes can be known with near certainty.

A phenomenon is random if individual outcomes are uncertain, but pattern from many individual outcomes is known. It is assumed that the phenomenon can be repeated indefinitely under essentially the same conditions.

Examples of random phenomenon include tossing coins, dice, and cards, and we can use a computer to simulate many repetitions of an event, like tossing a coin or rolling dice.

Example: Rolling Fair Dice

If we roll a single die, we can expect to get one of six possible outcomes: 1,2,3,4,5,6.

This set of all possible outcomes is the sample space S.

An event is the outcome of a random phenomenon, in this example, rolling the die.

A probability model is a description of a random phenomenon consisting of

- the sample space S,
- a way of assigning probabilities to an event.

If we assume the die is fair, that is, all outcomes are equally likely, then the probability model for rolling a single die consists of the sample space $S = \{1, 2, 3, 4, 5, 6\}$ where each element of the sample space has probability $1/6 \sim 0.16666667$.

Event	1	2	3	4	5	6
Probability	1/6	1/6	1/6	1/6	1/6	1/6

Let $P(A)$ represent the probability that event A occurs. For this case, we can write $P(6) = 1/6$, which reads "the probability of rolling a six is one sixth". Since the die is fair, we can write $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$.

Go to The Dice Experiment (http://www.math.uah.edu/stat/prob/index.xml): Update 10, Stop 1000, single Die, Show Y.

- The probability of rolling any of the six possibilities should be $1/6 \sim 0.1666777$.
- (the number of times we rolled a one)/(number of rolls $\sim 1/6$.
- We approached this probability, but did not get to it exactly, by taking more trials.
- With more trials we would get closer to this probability.

Change to two dice in the Dice Experiment, notice that the probabilities are not all the same.

For two dice, there are 36 possible ways to roll the dice (see Fig 7.2 in text), all with equal probability since each of the die are fair:

$\boxed{1}\boxed{1}$	$\boxed{1}\boxed{2}$	$\boxed{1}\boxed{3}$	$\boxed{1}\boxed{4}$	$\boxed{1}\boxed{5}$	$\boxed{1}\boxed{6}$
$\boxed{2}\boxed{1}$	$\boxed{2}\boxed{2}$	$\boxed{2}\boxed{3}$	$\boxed{2}\boxed{4}$	$\boxed{2}\boxed{5}$	$\boxed{2}\boxed{6}$
$\boxed{3}\boxed{1}$	$\boxed{3}\boxed{2}$	$\boxed{3}\boxed{3}$	$\boxed{3}\boxed{4}$	$\boxed{3}\boxed{5}$	$\boxed{3}\boxed{6}$
$\boxed{4}\boxed{1}$	$\boxed{4}\boxed{2}$	$\boxed{4}\boxed{3}$	$\boxed{4}\boxed{4}$	$\boxed{4}\boxed{5}$	$\boxed{4}\boxed{6}$
$\boxed{5}\boxed{1}$	$\boxed{5}\boxed{2}$	$\boxed{5}\boxed{3}$	$\boxed{5}\boxed{4}$	$\boxed{5}\boxed{5}$	$\boxed{5}\boxed{6}$
$\boxed{6}\boxed{1}$	$\boxed{6}\boxed{2}$	$\boxed{6}\boxed{3}$	$\boxed{6}\boxed{4}$	$\boxed{6}\boxed{5}$	$\boxed{6}\boxed{6}$

The sample space is $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and the probabilities are different for each element in the sample space because there are different ways to roll the elements in the sample space.

The event of rolling a 3, call it A, is possible in the following two ways:

$$A = \left\{\boxed{1}\boxed{2}, \boxed{2}\boxed{1}\right\}$$

So $P(A) = 2/36 = 1/18 \sim 0.055556$ assuming the dice are fair.

The event of rolling a 6, call it B, is possible in the five following ways:

$$B = \left\{\boxed{1}\boxed{5}, \boxed{2}\boxed{4}, \boxed{3}\boxed{3}, \boxed{4}\boxed{2}, \boxed{5}\boxed{1}\right\}$$

So $P(B) = 5/36 \sim 0.138889$ assuming the dice are fair.

The works because the dice were assumed to be fair, and the probability of rolling any of the 36 possibilities was the same.

How can we determine the probabilities of the outcomes when we are rolling two dice in a more mathematical manner, that will work if dice are not fair? We need to be able to combine the probabilities for single die in a systematic way.

Probability Rules

1. **Any probability is a number between 0 and 1.** If A is an event in the sample space S: $0 \leq P(A) \leq 1$. An even with 0 probability never occurs; an event with probability 1 occurs on every trial.
2. **All possible outcomes together must have probability 1.** If S is the sample space, then $P(S) = 1$.
3. **For example, the probability that an event does not occur is 1 minus the probability that an event does occur.** $P(A \text{ does not occur}) = 1 - P(A)$.
4. **Two events A and B are disjoint if they have no outcomes in common and so can never occur simultaneously.** If A and B are disjoint, then $P(A \text{ or } B) = P(A) + P(B)$.
5. **If two events A and B are independent, then the probability of both events is the product of the probabilities for each event**. $P(A \text{ and } B) = P(A)P(B)$.

Example: Rolling Two Unfair Dice (see text for parallel discussion on fair dice)

A pair of dice have been altered so they are not fair. Each has been altered so that

- 1 and 6 have probability 1/4 each, and 2,3,4,5 have probability 1/8.

For a single die, the probability rules are:

1. Satisfied, since each probability is between 0 and 1.
2. Satisfied, since the sum of all the probabilities are $1/4 + 1/8 + 1/8 + 1/8 + 1/8 + 1/4 = 1$.
3. For example, the probability that we don't roll a six is 1 minus the probability that we do roll a six $= 1 - 1/4 = 3/4 = 0.75$.
4. The probability that we roll a 1 or a 3 is
$$P(1 \text{ or } 3) = P(1) + P(3) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8} = 0.375$$
5. Since the roll of a single die is an independent event, previous rolls will not affect the outcome of the current roll.

Go back to Web site (The Dice Experiment) and see this (click on die) corresponds to 1-6 flat. Change to two dice to see two dice probabilities.

For two dice, the sample space is $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

The probability of rolling a 2 is:

$$\begin{aligned} P(2) &= P\left(\boxed{1}\boxed{1}\right) \\ &= P\left(\boxed{1}\right) P\left(\boxed{1}\right) \text{ the roll of the two dice are independent of each other} \\ &= \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} = 0.0625 \end{aligned}$$

Similarly,

$$\begin{aligned} P(\text{roll a } 3) &= P\left(\boxed{1}\boxed{2} \text{ or } \boxed{2}\boxed{1}\right) \\ &= P\left(\boxed{1}\boxed{2}\right) + P\left(\boxed{2}\boxed{1}\right) \text{ rule 4.} \\ &= P(\boxed{1})P(\boxed{2}) + P(\boxed{2})P(\boxed{1}) \text{ rule 5.} \\ &= \frac{1}{4} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{16} = 0.0625 \end{aligned}$$

In this way we can calculate the probability model for rolling two unfair dice:

Event	2	3	4	5	6	7	8	9	10	11	12
Probability	1/16	1/16	5/64	3/32	7/64	3/16	7/64	3/32	5/64	1/16	1/16

We have constructed a probability model for a finite sample space, which is assigning a probability for each individual outcome in the sample space. The probabilities must be numbers between 0 and 1, and they must sum to 1.

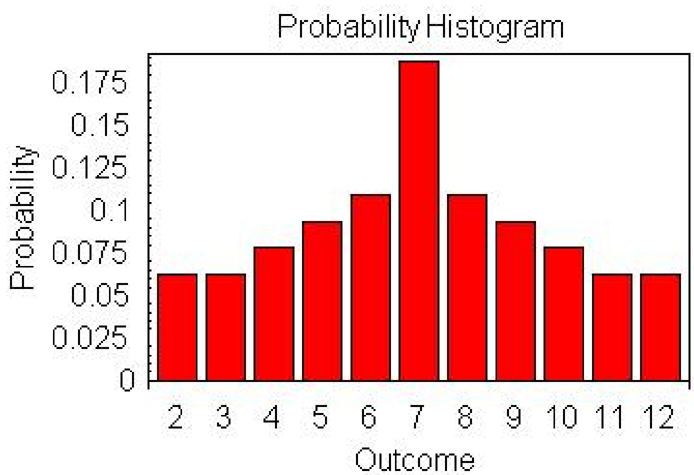
The probability of any event is the sum of the probabilities making up the event.

For example, the probability that rolling two unfair dice with probability model given above and getting an even number is

$$\begin{aligned} P(\text{outcome is even}) &= P(2)+P(4)+P(6)+P(8)+P(10)+P(12) \\ &= \frac{1}{16}+\frac{5}{64}+\frac{7}{64}+\frac{7}{64}+\frac{5}{64}+\frac{1}{16} \\ &= \frac{1}{2} \end{aligned}$$

What you have been looking at on the web (The Dice Experiment) is a probability histogram, where the height of each bar shows the probability of the outcome at the base. The heights of the bars all add to 1 by Rule 1 of the probability rules.

Here is the probability histogram for the rolling two unfair dice probability model:



We would not need Rule 5 if the dice were fair, since then we would know the probability of each 36 possible outcomes was the same. Since the text dealt with fair dice, they did not include Rule 5 in their list.

We used Rule 5 when we said the roll of the two dice, although done at the same time, were independent events. Rule 5 is also useful when events are performed in sequence, since then the events are independent.

For example, what is the probability of rolling a single fair die 5 times and getting a 4 each time? The answer is

$$\begin{aligned} P(\text{getting 5 straight rolls of 4}) &= P\left(\boxed{4}\boxed{4}\boxed{4}\boxed{4}\boxed{4}\right) \\ &= P\left(\boxed{4}\right)P\left(\boxed{4}\right)P\left(\boxed{4}\right)P\left(\boxed{4}\right)P\left(\boxed{4}\right) \\ &= \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} \\ &= \frac{1}{7776} \sim 0.00013 \end{aligned}$$

which tells us it would happen about 1 time out of 10 thousand times you tried!

Mean of a Probability Model

We can find the mean of a probability model in the following manner. The mean can be thought of as the weighted average of the outcomes.

Suppose the sample space S has outcomes x_i with probabilities p_i. The mean of the probability distribution is given the greek symbol "mu"

$$\text{mean } = \mu = x_1p_1 + x_2p_2 + \cdots + x_kp_k.$$

Example At a store, the number of people in checkout lines varies. The probability model for the number of people in a randomly chosen line is

Number in line	0	1	2	3	4	5
Probability	0.08	0.15	0.20	0.22	0.15	0.20

What is the mean number of people in a line?

$$\begin{aligned} \text{mean number people in line} &= 0(0.08) + 1(0.15) + 2(0.20) + 3(0.22) + 4(0.15) + 5(0.20) \\ &= 2.81 \end{aligned}$$

Example A fair die is rolled. If a number 1 or 2 appears, you will receive \$ 5. If any other number appears, you will lose \$2. What is the mean value of one trial of this game?

Fair dice means each face has a probability of 1/6.

$$\begin{aligned} \text{mean} &= \$5(1/6) + \$5(1/6) - \$2(1/6) - \$2(1/6) - \$2(1/6) - \$2(1/6) \\ &= \$\frac{1}{3} \end{aligned}$$

The Law of Large Numbers

We have already seen this in action, when we looked at how the long run trials of rolling dice lead to proportions which were close to 1/6 for all faces. This is the law of large numbers, which states that a random phenomenon repeated a large number of times will

- have proportion of trials on which each outcome occurs gets closer and closer to the probability of that outcome, and
- the mean $\bar{x}$ of the outcomes gets closer and closer to the mean of the probability model μ.