**Concepts:** Solving Linear Equations, Identities, Conditional Equations, Inconsistant Equations, Solving Rational Equations, Solving Basic Absolute Value Equations

### Solving Linear Equations

An *equation* involves an equal sign and indicates that two expressions have the same value.

$x + 42 = 67(4 - x)$ is an equation, and means $x + 42$ has the same value as $67(4 - x)$.

*Equivalent equations* are equations that have exactly the same solution.

Solving an equation typically involves using the rules of algebra to construct a series of equivalent equations until you determine a numerical solution for an unknown variable. When using algebra, show enough intermediate steps in your solution to get the correct answer. A good rule of thumb is to write enough so that a classmate could read your solution and understand all the steps you used without having you explain it to them.

**The Addition Property of Equality:** If the same number is added to both sides of an equation, the results on both sides are equal in value (you have constructed an equivalent equation).

$$x + 42 = 67(4 - x) \text{ is an equation,}$$
$$x + 42 + 76 = 67(4 - x) + 76 \text{ is an equivalent equation.}$$

There is also a Subtraction Property of Equality, but since you could think of "subtraction" as "adding the negative", I am not including it here since it is the same as the Addition Property of Equality.

**The Multiplication Property of Equality:** If both sides of an equation are multiplied by the same nonzero number, the results on both sides are equal in value (you have constructed an equivalent equation).

$$x + 42 = 67(4 - x) \text{ is an equation,}$$
$$132(x + 42) = 132 \times 67(4 - x) \text{ is an equivalent equation.}$$

**The Division Property of Equality:** If both sides of an equation are divided by the same nonzero number, the results on both sides are equal in value (you have constructed an equivalent equation).

$$x + 42 = 67(4 - x) \text{ is an equation,}$$
$$\frac{(x + 42)}{69} = \frac{67(4 - x)}{69} \text{ is an equivalent equation.}$$

Note that you have to be careful with division and multiplication. Make sure you multiply or divide each entire side of the equation–if you don't, you will be making an algebra error!

**Solving an equation of the form** $ax + b = cx + d$ (or even slightly more complicated equations) involves constructing a series of equivalent equations that ends with the equivalent equation $x =$ a number. The following steps are required:

1. Clear any parentheses, and simply as much as possible (simplifying is just advice to make things easier).
2. Collect like terms using the Addition Property of Equality if necessary.
3. Isolate the variable term.
4. Use the Division Property of Equality to isolate the variable.

5. Check your answer by substituting back in the original equation to see if your answer is correct.

An *identity* is an equation that is satisfied by every real number for which both sides of the equation are defined.

A *conditional equation* is an equation that is satisfied by at least one real number, but is not an identity. The set of values that satisfy the conditional equation is called the *solution set.*

An *inconsistant equation* is an equation that has no solutions. Inconsistant equations are equivalent to a False statement, which you would see if you tried to solve for the variable.

**Example** Solve $\frac{2}{3}(x+4) = 6 - \frac{1}{4}(3x-2) - 1$.

Remember, you might choose a different route to the solution that is entirely correct.
The goal is first to isolate a single term with $x$ in it on one side of the equation.

$$\frac{2}{3}(x+4) = 6 - \frac{1}{4}(3x-2) - 1$$

$$\frac{2}{3}x + \frac{8}{3} = 6 - \frac{3}{4}x + \frac{2}{4} - 1 \text{ (I choose to clear parentheses first)}$$

$$\frac{2}{3}x + \frac{8}{3} = 6 - \frac{3}{4}x + \frac{2}{4} - 1 \text{ (simplify on each side of equal side by collecting like terms)}$$

$$\frac{2}{3}x + \frac{8}{3} = \frac{11}{2} - \frac{3}{4}x$$

(use Addition Principle to move all terms with $x$ to left side, all other terms to right side)

$$\frac{2}{3}x + \frac{3}{4}x + \frac{8}{3} - \frac{8}{3} = \frac{11}{2} - \frac{8}{3} - \frac{3}{4}x + \frac{3}{4}x$$

$$\frac{2}{3}x + \frac{3}{4}x = \frac{11}{2} - \frac{8}{3} \text{ (now collect like terms)}$$

$$\frac{17}{12}x = \frac{17}{6} \text{ (now use Multiplication Principle to isolate the } x\text{)}$$

$$\frac{12}{17} \times \frac{17}{12}x = \frac{12}{17} \times \frac{17}{6} \text{ (simplify)}$$

$$x = 2$$

**Complex Rational Expressions and Rational Equations**

When **solving equations involving rational expressions,** the following technique always works:

1. Determine the LCD of all the denominators.
2. Multiply each term in the equation by the the LCD.
3. Solve the resulting equation.
4. Check the solution–you should exclude any solution that you find which makes the LCD zero (it would result in division by zero in the original equation, so it is not allowed). These excluded solutions are called <u>extraneous solutions</u>.

If you have a **complex rational expression** (ie., you have multiple denominators), get a common denominator in the overall numerator and denominator, then use the rules for division to simplify. An example helps make the process clear.

$$
\begin{aligned}
\frac{\frac{1-x}{x}+\frac{x}{y}}{\frac{1}{x^2y}-\frac{y}{x}} &= \frac{\frac{1-x}{x}\cdot\frac{y}{y}+\frac{x}{y}\cdot\frac{x}{x}}{\frac{1}{x^2y}-\frac{y}{x}\cdot\frac{xy}{xy}} \\
&= \frac{\frac{(1-x)y}{xy}+\frac{x^2}{xy}}{\frac{1}{x^2y}-\frac{xy^2}{x^2y}} \\
&= \frac{\left(\frac{y-xy+x^2}{xy}\right)}{\left(\frac{1-xy^2}{x^2y}\right)} \\
&= \left(\frac{y-xy+x^2}{\cancel{xy}}\right)\cdot\left(\frac{x^{\cancel{2}}\cancel{y}}{1-xy^2}\right) \\
&= \left(y-xy+x^2\right)\cdot\left(\frac{x}{1-xy^2}\right) \\
&= \frac{(y-xy+x^2)x}{1-xy^2}
\end{aligned}
$$

**Example** Solve $\dfrac{x+11}{x^2-5x+4}+\dfrac{3}{x-1}=\dfrac{5}{x-4}$.
Factor $x^2-5x+4$: Need two numbers whose product is 4 and sum is $-5$: $-4,-1$.

$$x^2-5x+4=(x-4)(x-1).$$

The LCD for the the equation is $(x-4)(x-1)$. Multiply all terms in the equation by this LCD:

$$
\begin{aligned}
&\frac{x+11}{x^2-5x+4}+\frac{3}{x-1}=\frac{5}{x-4} \\
&\frac{x+11}{\cancel{(x-4)(x-1)}}\cdot\cancel{(x-4)(x-1)}+\frac{3}{\cancel{x-1}}\cdot(x-4)\cancel{(x-1)}=\frac{5}{\cancel{x-4}}\cdot\cancel{(x-4)}(x-1) \\
&x+11+3(x-4)=5(x-1) \\
&x+11+3x-12=5x-5 \\
&4x-1=5x-5 \\
&4x-5x=-5+1 \\
&-x=-4 \\
&x=4
\end{aligned}
$$

We aren't done until we verify this is actually a solution. Since $x=4$ makes the LCD zero, this is <u>not</u> a solution since it would result in division by zero.
Therefore, $x=4$ is an extraneous solution (meaning it is not a solution), and the original equation has no solution.

**Solving Basic Absolute Value Equalities**

For equalities of the form $\left|ax+b\right| = k$ where $k > 0$, the solution is

$$ax + b = k \qquad \text{or} \qquad ax + b = -k.$$

The solution set will consist of two distinct numbers, $\{(k-b)/a, (-k-b)/a\}$.

For equalities of the form $\left|ax+b\right| = 0$, the solution is

$$ax + b = 0.$$

The solution set will be one distinct number, $\{-b/a\}$.

For equalities of the form $\left|ax+b\right| = k$ where $k < 0$, there is no solution. The solution will be the empty set, $\emptyset$.

**Example** Solve the inequality $\left|3x+2\right| = 24$.

$$\begin{aligned} \left|3x+2\right| &= 24 \\ 3x+2 = 24 \text{ or } 3x+2 &= -24 \\ 3x = 22 \text{ or } 3x &= -26 \\ x = \frac{22}{3} \text{ or } x &= -\frac{26}{3} \end{aligned}$$