Concepts: Pythagorean Formula, Distance Formula, The Circle, Completing the Square, Graphing Lines

The distance between two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The distance formula is based on the Pythagorean Theorem.

The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1,+y_2}{2}\right)$. There is a very nice proof of this in the text, you should make sure to understand that proof.

The Circle

A *circle* is a set of all the points in a plane that lie a fixed distance from a given point in the plane.

If we want the equation of the circle that represents the set of points a distance r from (h, k), we can use the distance formula:

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

The *standard form* for the equation of a circle with center (h, k) and radius $r > 0$ is

$$(x-h)^2 + (y-k)^2 = r^2.$$

Note that squaring both sides of an equation is valid only if both sides of the equation are positive, which is where the restriction $r > 0$ came from.

If you can write a circle in the standard form, it is easy to graph the circle. If the circle is not in standard form, you can use completing the square to write it in standard form.

Completing the Square

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c && \text{Factor so the coefficient of } x^2 \text{ is 1. Coefficient of } x \text{ is } \tfrac{b}{a} \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c && \text{the blue terms add to zero; we haven't changed the equality} \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c && \text{the red terms can be collected together as a perfect square} \\
&= a\left(\left[x + \frac{b}{2a}\right]^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left[x + \frac{b}{2a}\right]^2 - a\left(\frac{b}{2a}\right)^2 + c && \text{simplify} \\
&= a\left[x + \frac{b}{2a}\right]^2 - \frac{b^2}{4a} + c \\
&= a\left[x + \frac{b}{2a}\right]^2 - \frac{b^2 - 4ac}{4a}
\end{aligned}$$

Here is the process of completing the square in words:

- Factor so that there is just a 1 in front of the x^2 term.
- Identify the coefficient of the x term.
- Take half of this coefficient and square, then add and subtract so you don't change the equation.
- Fold up the perfect square you have created.
- Simplify.

Example Determine the center and radius of the circle $x^2 + y^2 = 4x + 3y$.
We will need to complete the square in both x and y.

$$
\begin{aligned}
x^2 + y^2 &= 4x + 3y \\
x^2 - 4x + y^2 - 3y &= 0 \text{ note it is important to have coefficient 1 for the } x^2 \text{ and } y^2 \\
x^2 - 4x + 4 - 4 + y^2 - 3y + \frac{9}{4} - \frac{9}{4} &= 0 \\
(x^2 - 4x + 4) + (y^2 - 3y + \frac{9}{4}) &= +4 + \frac{9}{4} \\
(x-2)^2 + (y - \frac{3}{2})^2 &= \frac{25}{4} \\
(x-2)^2 + (y - \frac{3}{2})^2 &= \left(\frac{5}{2}\right)^2
\end{aligned}
$$

So the circle has center $(2, 3/2)$ and radius $5/2$.

Completing the square can also be used to solve quadratic equations.

Example Use completing the square to solve $2x^2 + 4x + 1 = 0$.

$$2x^2 + 6x + 1 = 0$$

We MUST have a coefficient of 1 in front of the x^2 before we complete the square.

$$x^2 + 3x + \frac{1}{2} = 0$$

$$x^2 - 3x + \frac{1}{2} = 0 \text{ To complete the square: } \left(\frac{3}{2}\right)^2.$$

$$\underline{x^2 + 3x + \left(\frac{3}{2}\right)^2} - \left(\frac{3}{2}\right)^2 + \frac{1}{2} = 0 \text{ underlined piece is a perfect square}$$

$$\underline{\left(x + \frac{3}{2}\right)}^2 = \frac{9}{4} - \frac{1}{2}$$

$$\underline{\left(x + \frac{3}{2}\right)}^2 = \frac{7}{4}$$

$$x + \frac{3}{2} = \pm\sqrt{\frac{7}{4}}$$

$$x = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}$$

The Line

If A, B, and C are real numbers, then the graph of the equation

$Ax + By = C$ (standard form)

is a straight line, provided A and B are not both zero.

The standard form is particularly useful, since it contains not only lines of the form $y = mx + b$ (which we will see in Section 1.4) but also horizontal and vertical lines.

The x-intercept is where the line crosses the x-axis, and the y-intercept is where the line crosses the y-axis.

You can graph any line by finding two points on the line and drawing the line through those two points.