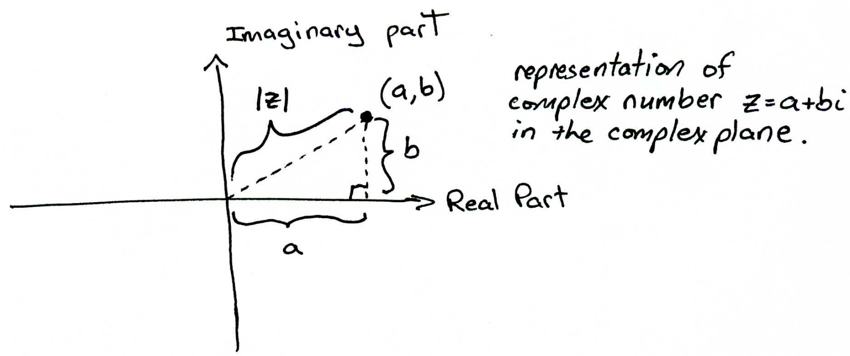
Concepts: basics of complex numbers

Complex Numbers

A complex number is typically denoted $z \in \mathbb{C}$. We write the complex number in terms of its real part a and imaginary part b as $z = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

Two complex numbers are equal if and only if their real and imaginary parts are equal.

A complex number can be represented as a point on the *complex plane*:



Notice that if $a \neq 0, b = 0$, the number a is a real number (as it lies on the real axis). *Therefore, all real numbers are also complex numbers.*

If $a = 0, b \neq 0$, the number bi is called an imaginary number (as it lies on the imaginary axis).

The **complex conjugate** of $z = a + bi$ is defined as $\bar{z} = a - bi$.

The magnitude of a complex number $z = a + bi$ is given by the formula

$$\begin{aligned} |z| &= \sqrt{z\bar{z}} \\ &= \sqrt{(a+bi)(a-bi)} \\ &= \sqrt{a^2+b^2} \end{aligned}$$

Note: $|z|$ reduces to the familiar absolute value if $z \in \mathbb{R}$.

Arithmetic with Complex Numbers

Addition and subtraction: collect real and imaginary parts:

$$(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$$

Multiplication: use the distributive property and the fact $i^2 = -1$ (do not memorize this formula):

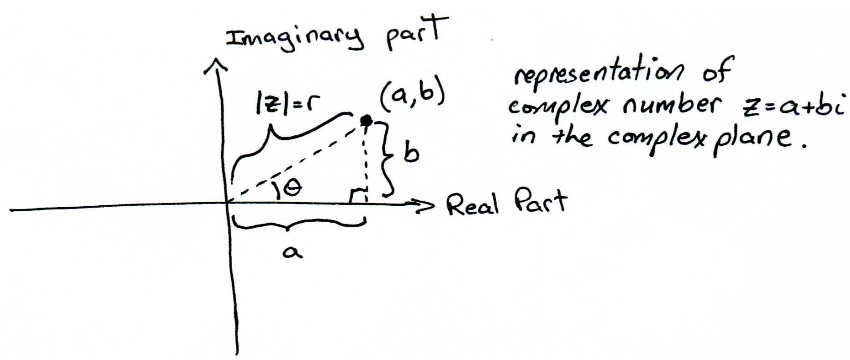
$$\begin{aligned} (a+bi)(c+di) &= a(c+di) + bi(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Division: multiply the numerator and denominator by the complex conjugate of the denominator (do not memorize this formula):

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}\\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)}\\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\\ &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i\end{aligned}$$

Aside and Looking Ahead: Connection of Complex Numbers with Trigonometry and Exponentials

Looking ahead, there is a deep and beautiful connection between complex numbers, trigonometry, and exponentials. Looking back at our diagram of the complex plane, we can introduce an angle θ and distance $r = |z|$ from polar coordinates.



This allows us to write

$$\begin{aligned}z &= a+bi\\ &= |z|\cos\theta + |z|\sin\theta i\\ &= |z|(\cos\theta + i\sin\theta)\end{aligned}$$

Looking ahead even further, there is a result that can be proven using calculus known as Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$. This means that we can write complex numbers in a variety of ways:

$$z = a+bi = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$$

These powerful representations can be used to prove trig identities, determine roots of complex numbers, solve linear differential equations, and a host of other things.

In my opinion, Euler's formula is one of the most amazing formulas in mathematics. It kicks ASS!!