Concepts: Solving Quadratic Equations, Completing the Square, The Quadratic Formula.

Solving Quadratic Equations

The *square root property:* For any real number k, the equation $x^2 = k$ is equivalent to $x = \pm\sqrt{k}$.

You typically have three options to solve a quadratic equation of the form $ax^2 + bx + c = 0$:

1. factor
2. complete the square
3. use the quadratic formula

Factoring only works if you can factor the quadratic (obviously) and relies on the *zero factor property*: If A and B are alegbraic expressions, then the equation $AB = 0$ is equivalent to the compound statement $A = 0$ or $B = 0$.

We have already seen that completing the square is what leads to the quadratic formula.

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c && \text{Factor so the coefficient of } x^2 \text{ is 1. Coefficient of } x \text{ is } \tfrac{b}{a} \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c && \text{the blue terms add to zero; we haven't changed the equality} \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c && \text{the red terms can be collected together as a perfect square} \\
&= a\left(\left[x + \frac{b}{2a}\right]^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left[x + \frac{b}{2a}\right]^2 - a\left(\frac{b}{2a}\right)^2 + c && \text{simplify} \\
&= a\left[x + \frac{b}{2a}\right]^2 - \frac{b^2}{4a} + c \\
&= a\left[x + \frac{b}{2a}\right]^2 - \frac{b^2 - 4ac}{4a}
\end{aligned}$$

This is how you derive the quadratic formula, since from here we set this equal to zero and solve for x:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
a\left[x + \frac{b}{2a}\right]^2 - \frac{b^2 - 4ac}{4a} &= 0 \\
a\left[x + \frac{b}{2a}\right]^2 &= \frac{b^2 - 4ac}{4a} \\
\left[x + \frac{b}{2a}\right]^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \text{ use the square root property} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

For simpler quadratic equations of the form $ax^2 + c = 0$ we can use the square root property as well as the above techniques. Notice if $c > 0$ then we have complex values solutions.

Quadratic Formula

Memorize: The two solutions to the equation $ax^2 + bx + c = 0$ are given by the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You can use the quadratic formula to solve quadratic equations that previously you solved by factoring.

Example Solve $\frac{1}{15} + \frac{3}{y} = \frac{4}{y+1}$.

$$\begin{aligned}
\frac{1}{15} + \frac{3}{y} &= \frac{4}{y+1} \text{ (multiply by the LCD which is } 15y(y+1)) \\
\frac{1}{\cancel{15}} \cdot \cancel{15}y(y+1) + \frac{3}{\cancel{y}} \cdot 15\cancel{y}(y+1) &= \frac{4}{\cancel{y+1}} \cdot 15y\cancel{(y+1)} \text{ (simplify)} \\
y(y+1) + 45(y+1) &= 60y \\
y^2 + y + 45y + 45 &= 60y \\
y^2 - 14y + 45 &= 0 \text{ (use quadratic formula–I always write it out)} \\
y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
y &= \frac{14 \pm \sqrt{(-14)^2 - 4(1)(45)}}{2(1)} \\
y = \frac{14 \pm \sqrt{16}}{2} = \frac{14 \pm 4}{2} &= 7 \pm 2 = 9 \text{ or } y = 5
\end{aligned}$$

Since neither $y = 9$ nor $y = 5$ makes the LCD zero, these are both solutions.

Example Solve $(2x+7)^2 - \frac{3}{4} = 14$.

$$\begin{aligned}
(2x+7)^2 - \frac{3}{4} &= 14 \\
(2x+7)^2 &= 14 + \frac{3}{4} \\
(2x+7)^2 &= \frac{59}{4} \\
\sqrt{(2x+7)^2} &= \pm\sqrt{\frac{59}{4}} \\
2x+7 &= \pm\frac{\sqrt{59}}{2} \\
2x &= -7 \pm \frac{\sqrt{59}}{2} \\
x &= -\frac{7}{2} \pm \frac{\sqrt{59}}{4}
\end{aligned}$$

You have to check if these are solutions by substituting back:

$$\begin{aligned}
(2x+7)^2 - \frac{3}{4} &= \left(2\left(-\frac{7}{2} + \frac{\sqrt{59}}{4}\right) + 7\right)^2 - \frac{3}{4} \\
&= \left(-7 + \frac{\sqrt{59}}{4} + 7\right)^2 - \frac{3}{4} \\
&= \left(\frac{\sqrt{59}}{2}\right)^2 - \frac{3}{4} \\
&= \frac{59}{4} - \frac{3}{4} = \frac{56}{4} = 14
\end{aligned}$$

$$\begin{aligned}
(2x+7)^2 - \frac{3}{4} &= \left(2\left(-\frac{7}{2} - \frac{\sqrt{59}}{4}\right) + 7\right)^2 - \frac{3}{4} \\
&= \left(-7 - \frac{\sqrt{59}}{4} + 7\right)^2 - \frac{3}{4} \\
&= \left(-\frac{\sqrt{59}}{2}\right)^2 - \frac{3}{4} \\
&= \frac{59}{4} - \frac{3}{4} = \frac{56}{4} = 14
\end{aligned}$$

So both are solutions. $x = -\frac{7}{2} \pm \frac{\sqrt{59}}{4}$.