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Concepts: ellipses (center, vertices, foci, focal axis, Pythagorean relation, reflective property, sketching).

Definition of an Ellipse

Definition: An ellipse is the set of all points in a plane whose sum of their distances to two particular points (the foci) in the plane is constant. The line through the foci is the *focal axis*. The point midway between the foci is the *center*. The points where the ellipse intersects its focal axis are the *vertices*.

Let's derive the algebraic equation for an ellipse. Without loss of generality, we can assume the center of the ellipse is at the origin $(0,0)$, and the foci are at $F_1(-c,0)$ and $F_2(c,0)$. The vertices are at $(-a,0)$ and $(a,0)$, where $a > c$. The sum of the distance to the two foci will be a constant, and the constant must be $2a$ (think of the point P being on the x-axis to see this).
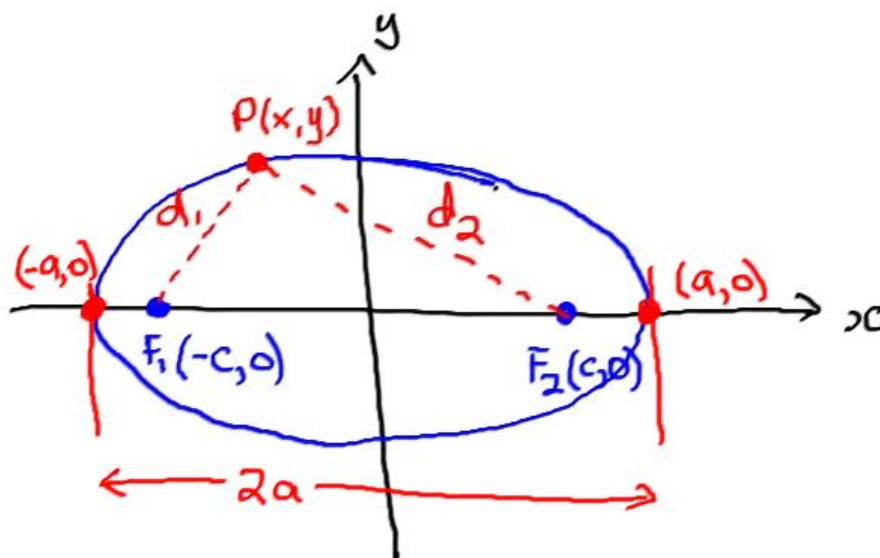
From the definition of ellipse, we must have for an arbitrary point $P(x,y)$ on the ellipse:

$$\text{distance to } F_1 + \text{distance to } F_2 = 2a$$
$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$
$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

The above is one representation of an ellipse, but not particularly useful. It would be nice to have a representation that did not rely on the two constants a and c. At the moment, we do not have a good understanding of what the quantity a represents (other than it is twice the sum of the distances to the foci). With the goal of a better representation in mind, we start manipulating the equation, and an obvious first step is to get rid of the square roots and see what happens:

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2} \quad \text{square}$$
$$(x+c)^2 + \cancel{y^2} = 4a^2 + (x-c)^2 + \cancel{y^2} - 4a\sqrt{(x-c)^2 + y^2}$$
$$(x+c)^2 - 4a^2 - (x-c)^2 = -4a\sqrt{(x-c)^2 + y^2}$$
$$\cancel{x^2} + \cancel{c^2} + 2xc - 4a^2 - \cancel{x^2} - \cancel{c^2} + 2xc = -4a\sqrt{(x-c)^2 + y^2}$$
$$4xc - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$
$$xc - a^2 = -a\sqrt{(x-c)^2 + y^2} \quad \text{square}$$
$$x^2c^2 + a^4 - 2xca^2 = a^2[(x-c)^2 + y^2]$$

Now that the square roots are gone, we look for ways to simplify this expression.

$$x^2c^2 + a^4 - 2xca^2 = a^2(x-c)^2 + a^2y^2$$
$$x^2c^2 + a^4 - 2xca^2 = a^2(x^2 + c^2 - 2xc) + a^2y^2$$
$$x^2c^2 + a^4 - \cancel{2xca^2} = a^2x^2 + a^2c^2 - \cancel{2xca^2} + a^2y^2$$
$$x^2c^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$
$$x^2c^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

This looks better. Collect the x and y together. We can define new constants if that helps (and it will).

$$x^2c^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$
$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$
$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2) \text{ Let } b^2 = a^2 - c^2 > 0 \text{ since } a > c$$
$$x^2b^2 + a^2y^2 = a^2b^2$$
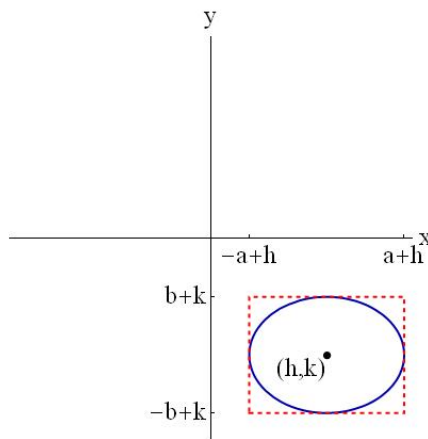$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the most useful expression for an ellipse. The relation $b^2 = a^2 - c^2 > 0$ is called the *Pythagorean relation* and is typically written as $c^2 = a^2 - b^2$, since it is most often used to determine the value of c, and hence the location of the foci.

The standard form for the equation of an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The transformed form is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ which puts the center of the ellipse at (h, k).
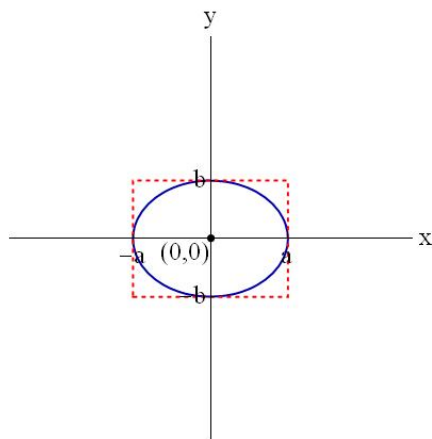
Standard Form: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Transformed Form: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

Notice that the ellipse is inside the box. Hyperbolas are created from the same box, but will be outside the box.

To sketch an ellipse, first sketch the box, then draw the ellipse inside.

You needn't memorize the numbers for the box, work it out each time:

For example, to sketch $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$:

Evaluate at $x = h \Rightarrow \dfrac{(y-k)^2}{b^2} = 1 \Rightarrow y = k \pm b$.

Evaluate at $y = k \Rightarrow \dfrac{(x-h)^2}{a^2} = 1 \Rightarrow x = h \pm a$.

This gives you the sides of the box.

Note: If $b > a$, then the focal axis will be parallel to the y-axis. This will be obvious when you sketch the ellipse, and the associated Pythagorean relation will be $c^2 = b^2 - a^2$ (you want c to be a real number, so that is what guides your choice of using $c^2 = a^2 - b^2$ (focal axis parallel to x-axis) or $c^2 = b^2 - a^2$ (focal axis parallel to y-axis).

Note on Formulas: There are many formulas associated with conic sections. I recommend knowing how to sketch parabolas, ellipses, and hyperbolas by hand by understanding the basic properties of each. For ellipses, you should definitely know how to determine the locations of the center, vertices, foci, and focal axis (this entails knowing the Pythagorean relation for ellipses). Eccentricity is useful, but I will not require you to memorize the formula for eccentricity.
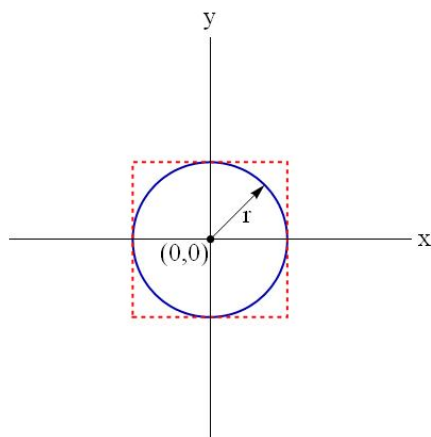
Circles

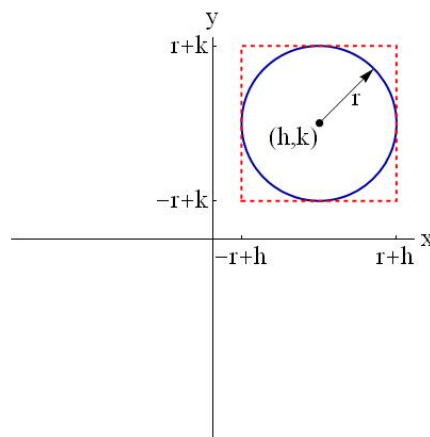Notice that an ellipse becomes a circle when $a = b = r$.

A *circle* is the set of all points in a plane equidistant from a particular point.

The standard form for the equation of a circle is $x^2 + y^2 = r^2$.

The transformed form is $(x - h)^2 + (y - k)^2 = r^2$.

Standard Form: $x^2 + y^2 = r^2$ Transformed Form: $(x - h)^2 + (y - k)^2 = r^2$
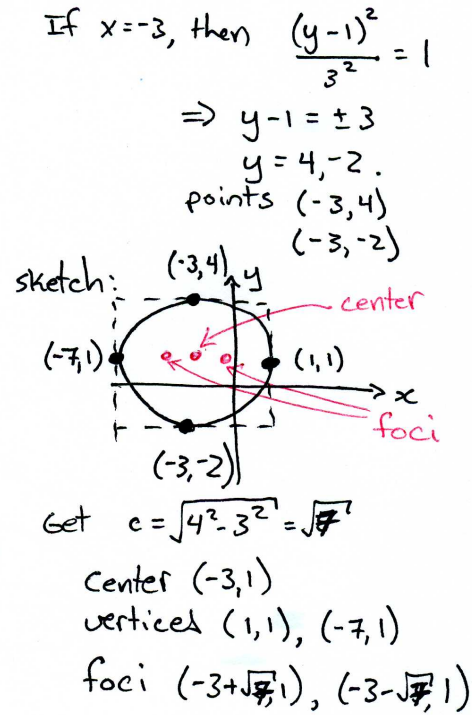
I draw the box around the circle since we will need one for ellipses and hyperbolas; it is not necessary to include the box for circles.

To sketch a circle, find h and k (you might have to complete the square in x and then again in y!) to get the center, and then sketch based on the transformed form.

The equation $x^2 + y^2 = r^2$ is called an *implicit function* since it implicitly defines two *explicit functions*, $y = f_1(x) = \sqrt{r^2 - x^2}$ and $y = f_2(x) = -\sqrt{r^2 - x^2}$ (the top and bottom of the circle separately pass the vertical line test we used to determine if we had an explicit function).

Example. Sketch by hand $9x^2 + 16y^2 + 54x - 32y - 47 = 0$. Locate the center, vertices, and foci of the ellipse.

$9x^2 + 16y^2 + 54x - 32y - 47 = 0$
complete the square in x and y
$9[\underbrace{x^2 + 6x + 9} - 9] + 16[\underbrace{y^2 - 2y + 1} - 1] = 47$

$9[(x+3)^2 - 9] + 16[(y-1)^2 - 1] = 47$

$9(x+3)^2 - 81 + 16(y-1)^2 - 16 = 47$

$9(x+3)^2 + 16(y-1)^2 = 144$

$\dfrac{(x+3)^2}{16} + \dfrac{(y-1)^2}{9} = 1$ ← divide by 144 to make this 1.

$\dfrac{(x+3)^2}{4^2} + \dfrac{(y-1)^2}{3^2} = 1$ ellipse!

Get Box: If $y = 1$, then $\dfrac{(x+3)^2}{4^2} = 1$

$\Rightarrow x + 3 = \pm 4$ points $(1,1)$
$\quad x = 1, -7$ $(-7,1)$

If $x = -3$, then $\dfrac{(y-1)^2}{3^2} = 1$

$\Rightarrow y - 1 = \pm 3$
$\quad y = 4, -2$.
points $(-3, 4)$
$\quad\quad (-3, -2)$

sketch:

Get $c = \sqrt{4^2 - 3^2} = \sqrt{7}$

Center $(-3, 1)$
vertices $(1,1)$, $(-7,1)$
foci $(-3+\sqrt{7}, 1)$, $(-3-\sqrt{7}, 1)$

Example. Sketch by hand $4x^2 + y^2 - 32x + 16y + 124 = 0$. Locate the center, vertices, and foci of the ellipse.

$4x^2 + y^2 - 32x + 16y + 124 = 0$
complete the square in x and y.
$4[\underbrace{x^2 - 8x + 16} - 16] + [\underbrace{y^2 + 16y + 64} - 64] = -124$

$4[(x-4)^2 - 16] + [(y+8)^2 - 64] = -124$

$4(x-4)^2 - 64 + (y+8)^2 - 64 = -124$

$4(x-4)^2 + (y+8)^2 = 4$

$(x-4)^2 + \dfrac{(y+8)^2}{2^2} = 1$ ellipse!

Get Box: if $x = 4$, then $\dfrac{(y+8)^2}{2^2} = 1$

$\Rightarrow y + 8 = \pm 2$
$\quad y = -6, -10$
points $(4, -6)$, $(4, -10)$

If $y = -8$, then $(x-4)^2 = 1$
$\Rightarrow x - 4 = \pm 1$
$\quad x = 5, 3$
points $(5, -8)$, $(3, -8)$.

sketch

Center $(4, -8)$. Vertices $(4, -6)$, $(4, -10)$
Get $c = \sqrt{2^2 - 1^2} = \sqrt{3}$
foci $(4, -8+\sqrt{3})$, $(4, -8-\sqrt{3})$.