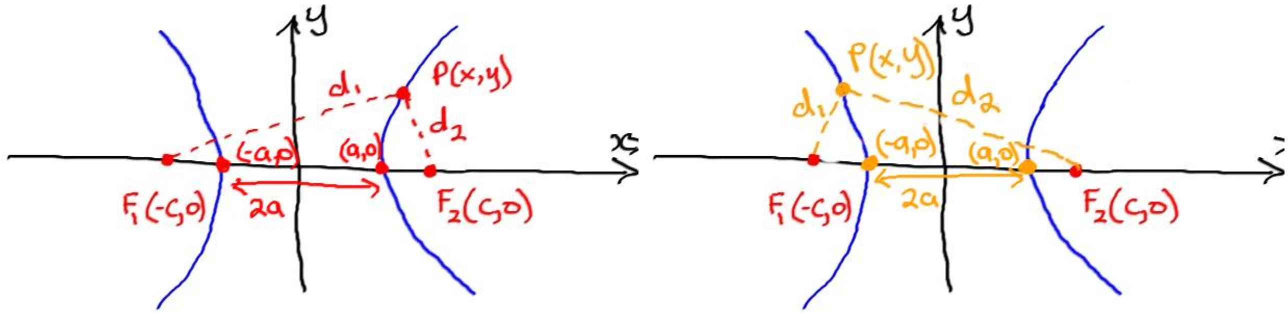
Concepts: hyperbolas (center, vertices, foci, focal axis, Pythagorean relation, reflective property, sketching).

Definition of an Hyperbola

Definition: An hyperbola is the set of all points in a plane whose distances from two particular points (the foci) in the plane have a constant difference. The line through the foci is the *focal axis*. The point midway between the foci is the *center*. The points where the hyperbola intersects its focal axis are the *vertices*.

Let's derive the algebraic equation for an hyperbola (the algebraic manipulations are very similar to what we did for an ellipse). Without loss of generality, we can assume the center of the hyperbola is at the origin $(0, 0)$, and the foci are at $F_1(-c, 0)$ and $F_2(c, 0)$. The vertices must be at $(-a, 0)$ and $(a, 0)$, where $c > a$.

Since we are dealing with a difference, there are two *branches* for the hyperbola: either $d_1 - d_2 = 2a$ or $d_2 - d_1 = 2a$ (think of the point P being on the x-axis to see this). These two cases can be combined as $d_1 - d_2 = \pm 2a$.



From the definition of hyperbola, we must have for an arbitrary point $P(x, y)$ on the hyperbola (either branch):

$$\begin{aligned}
\text{distance to } F_1 - \text{distance to } F_2 &= \pm 2a \\
\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} &= \pm 2a \\
\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= \pm 2a
\end{aligned}$$

The above is one representation of an hyperbola, but not particularly useful. It would be nice to have a representation that did not rely on the two constants a and c, and also not have the plus/minus. With the goal of a better representation in mind, we start manipulating the equation, and an obvious first step is to get rid of the square roots and see what happens.

$$\begin{aligned}
\sqrt{(x+c)^2 + y^2} &= \pm 2a + \sqrt{(x-c)^2 + y^2} \text{ square} \\
(x+c)^2 + \cancel{y^2} &= 4a^2 + (x-c)^2 + \cancel{y^2} \pm 4a\sqrt{(x-c)^2 + y^2} \\
(x+c)^2 - 4a^2 - (x-c)^2 &= \pm 4a\sqrt{(x-c)^2 + y^2} \\
\cancel{x^2} + \cancel{c^2} + 2xc - 4a^2 - \cancel{x^2} - \cancel{c^2} + 2xc &= \pm 4a\sqrt{(x-c)^2 + y^2} \\
4xc - 4a^2 &= \pm 4a\sqrt{(x-c)^2 + y^2} \\
xc - a^2 &= \pm a\sqrt{(x-c)^2 + y^2} \text{ square} \\
x^2c^2 + a^4 - 2xca^2 &= a^2[(x-c)^2 + y^2]
\end{aligned}$$

Now that the square roots are gone (and the plus/minus with them!), we look for ways to simplify this expression.

$$\begin{aligned}
x^2c^2 + a^4 - 2xca^2 &= a^2(x-c)^2 + a^2y^2 \\
x^2c^2 + a^4 - 2xca^2 &= a^2(x^2 + c^2 - 2xc) + a^2y^2 \\
x^2c^2 + a^4 - \cancel{2xca^2} &= a^2x^2 + a^2c^2 - \cancel{2xca^2} + a^2y^2 \\
x^2c^2 + a^4 &= a^2x^2 + a^2c^2 + a^2y^2 \\
x^2c^2 + a^4 &= a^2x^2 + a^2c^2 + a^2y^2
\end{aligned}$$

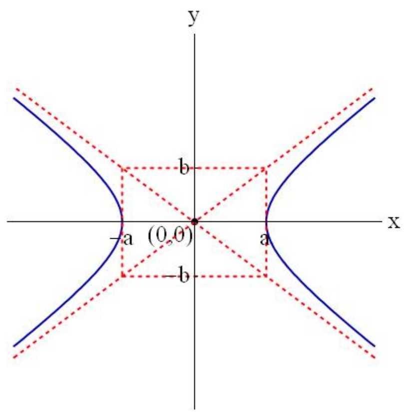
This looks better. Collect the x and y together. We can define new constants if that helps (and it will).

$$\begin{aligned} x^2c^2 - a^2x^2 - a^2y^2 &= a^2c^2 - a^4 \\ x^2(c^2 - a^2) - a^2y^2 &= a^2(c^2 - a^2) \text{ Let } b^2 = c^2 - a^2 > 0 \text{ since } c > a \\ x^2b^2 - a^2y^2 &= a^2b^2 \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \end{aligned}$$

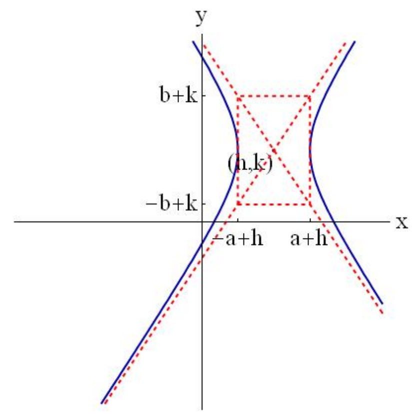
This is the most useful expression for an hyperbola. The relation $b^2 = c^2 - a^2 > 0$ is called the *Pythagorean relation* and is typically written as $c^2 = b^2 + a^2$, since it is most often used to determine the value of c, and hence the location of the foci. **Note that this is a different Pythagorean relation than the one for ellipses!**

The standard form for the equation of an hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$.

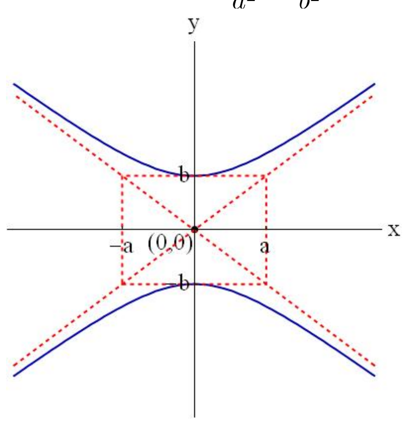
The transformed form is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ or $\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$.



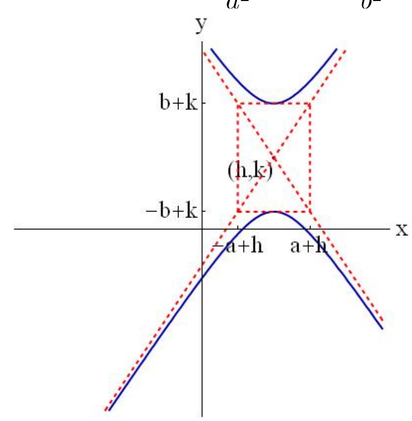
Standard Form: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



Transformed Form: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$



Standard Form: $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$



Transformed Form: $\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$

Notice the hyperbola is outside the box, and has slant asymptotes given by $y - k = \pm\dfrac{b}{a}(x - h)$.

To sketch a hyperbola, sketch the box as if you had an ellipse. Draw lines through the corners of the box.

Figure out if the hyperbola opens up/down or left/right by figuring out what the values are when $x = h$ and $y = k$. For example, for $\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$ when $y = k$ there is no solution to $-\dfrac{(x-h)^2}{a^2} = 1$, but when $x = h$, then $\dfrac{(y-k)^2}{b^2} = 1$ has two solutions, $y = b \pm k$, so the hyperbola must open up/down.

Note on Formulas: There are many formulas associated with conic sections. I recommend knowing how to sketch parabolas, ellipses, and hyperbolas by hand by understanding the basic properties of each. For hyperbolas, you should definitely know how to determine the locations of the center, vertices, foci, and focal axis (this entails knowing the Pythagorean relation for hyperbolas). Eccentricity is useful, but I will not require you to memorize the formula for eccentricity.

Example. Sketch by hand $25y^2 - 9x^2 - 50y - 54x - 281 = 0$. Locate the center, vertices, foci, and asymptotes of the hyperbola.

$25y^2 - 9x^2 - 50y - 54x - 281 = 0$

complete the square in x and y.

$25[y^2 - 2y + 1 - 1] - 9[x^2 + 6x + 9 - 9] = 281$

$25[(y-1)^2 - 1] - 9[(x+3)^2 - 9] = 281$

$25(y-1)^2 - 25 - 9(x+3)^2 + 81 = 281$

$25(y-1)^2 - 9(x+3)^2 = 225$

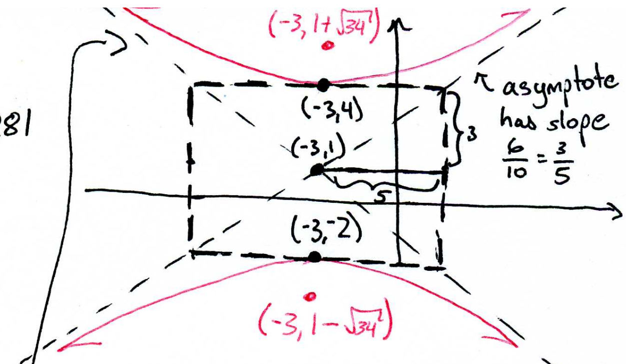
$\frac{(y-1)^2}{9} - \frac{(x+3)^2}{25} = 1$

$\frac{(y-1)^2}{3^2} - \frac{(x+3)^2}{5^2} = 1$ hyperbola!

Get Box: Center $(-3,1)$

x-width is $2 \cdot 5 = 10$

y-width is $2 \cdot 3 = 6$.



If $x = -3$, then $\frac{(y-1)^2}{3^2} = 1$

$\Rightarrow y - 1 = \pm 3$

$y = 4, -2$

vertices: $(-3,4)$ and $(-3,-2)$ (opens up/down)

Get $c = \sqrt{5^2 + 3^2} = \sqrt{34}$

foci $(-3, 1+\sqrt{34})$ and $(-3, 1-\sqrt{34})$.

asymptotes: $y - 1 = \pm \frac{3}{5}(x+3)$