Concepts: algebraic transformation of functions, graphical transformations of functions: shifts, stretches, reflections; Even/Odd/Neither.

Transformations of Functions

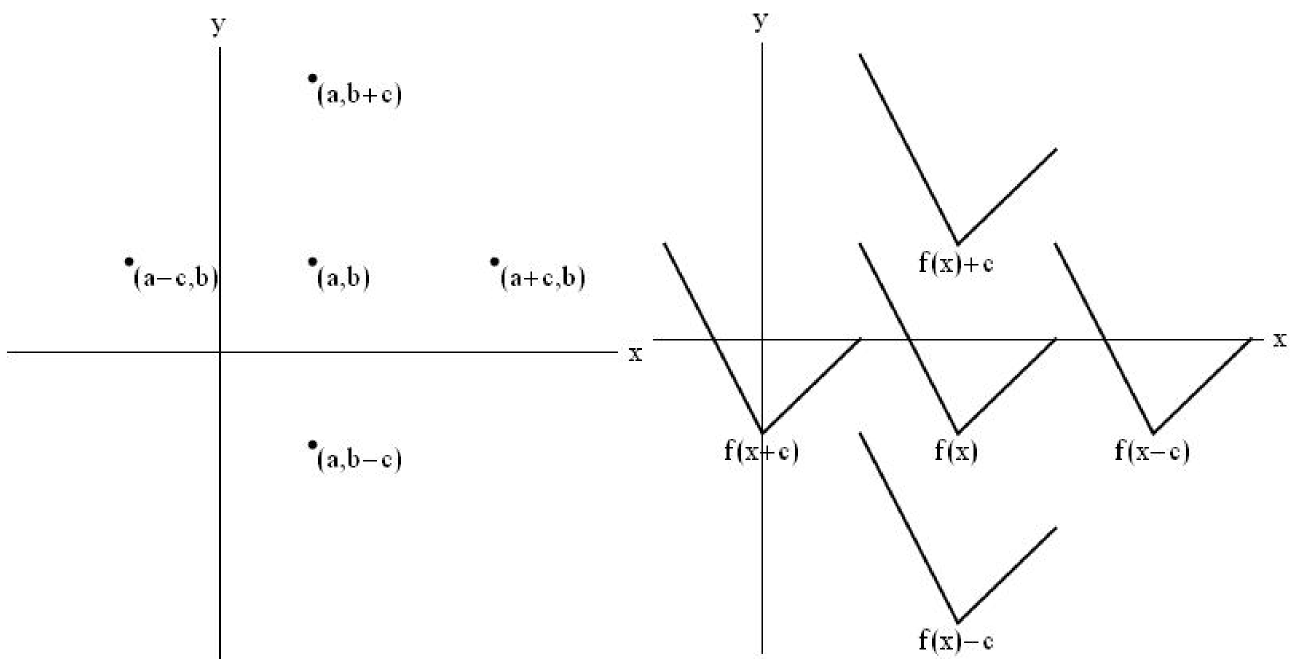
If we take a graph of a function and apply transformations to it, we can quickly determine the graph of a wide variety of related functions.

This is important since

- it allows us to obtain quickly sketches of complicated functions,
- it helps reinforce the properties of the basic functions from which the more complicated functions are built,
- gives us practice with functional notation.

Vertical and Horizontal Shifts:

Suppose $c > 0$. Here is how the ordered pair (a, b) is moved by adding or subtracting $c > 0$:



If $y = f(x)$ is our original function, the ordered pairs we have are $(x, y) = (x, f(x))$.

We can transform this function in the following ways:

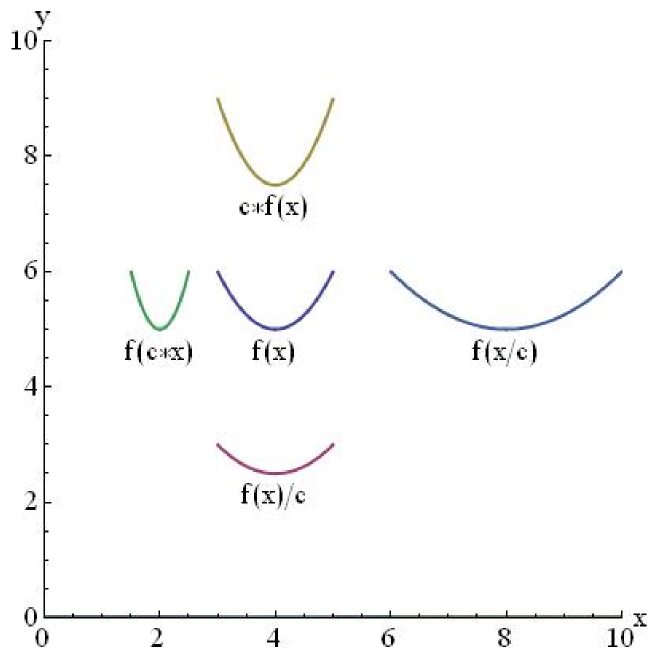
- A shift up of c units yields the ordered pairs $(x, f(x) + c)$ which means $y = f(x) + c$.
- A shift down of c units yields the ordered pairs $(x, f(x) - c)$ which means $y = f(x) - c$.
- A shift right of c units yields the ordered pairs $(x + c, f(x)) = (x, f(x - c))$ which means $y = f(x - c)$.
- A shift left of c units yields the ordered pairs $(x - c, f(x)) = (x, f(x + c))$ which means $y = f(x + c)$.

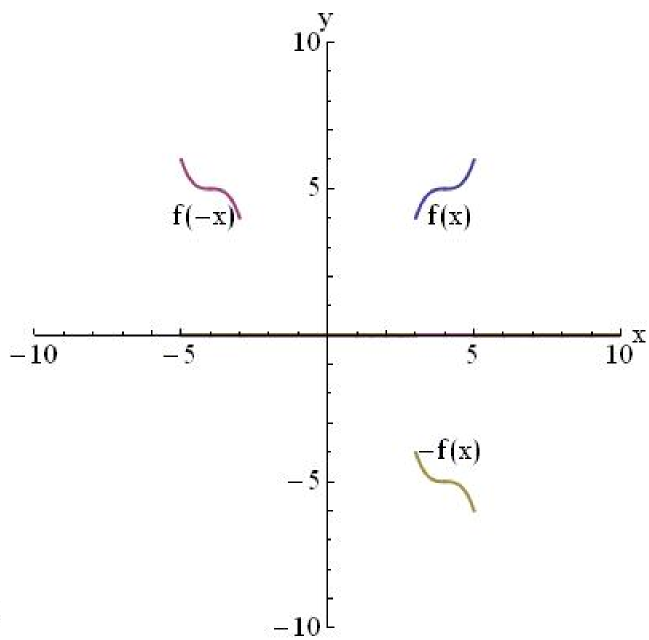
Vertical and Horizontal Stretching and Reflecting:

Stretching: If $c > 1$ then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction.

Reflection: The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis, because our ordered pair has changed from $(x, f(x))$ to $(x, -f(x))$.

Suppose $c > 1$. Then, to obtain the graph of
$y = cf(x)$ stretch the graph of $y = f(x)$ vertically by a factor of c,
$y = (1/c)f(x)$ compress the graph of $y = f(x)$ vertically by a factor of c,
$y = f(cx)$ compress the graph of $y = f(x)$ horizontally by a factor of c,
$y = f(x/c)$ stretch the graph of $y = f(x)$ horizontally by a factor of c,
$y = -f(x)$ reflect the graph of $y = f(x)$ about the x-axis,
$y = f(-x)$ reflect the graph of $y = f(x)$ about the y-axis.





NOTE: The stretching looks like translations, but notice how the shape of the graph changes. The translations and reflections are *rigid* transformation because the shape of the graph is not changed. Stretching and compressing are *nonrigid* transformations since the shape of the graph is changed.

You can combine these transformations, but be careful to remember that the order in which these transformations are performed can sometimes matter!

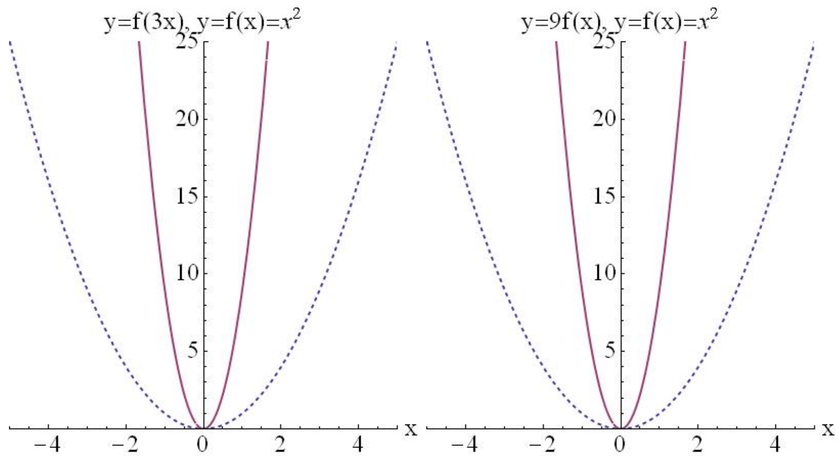
Key Ideas

Outside f is a vertical transformation.
Inside f is a horizontal transformation, and the action is the opposite of what it would be if it was outside f.

i.e. $f(x) + 3$ moves $f(x)$ three units up; $f(x + 3)$ moves $f(x)$ three units to left.
$3f(x)$ stretches $f(x)$ vertically by a factor of three; $f(3x)$ compresses $f(x)$ horizontally by a factor of three.

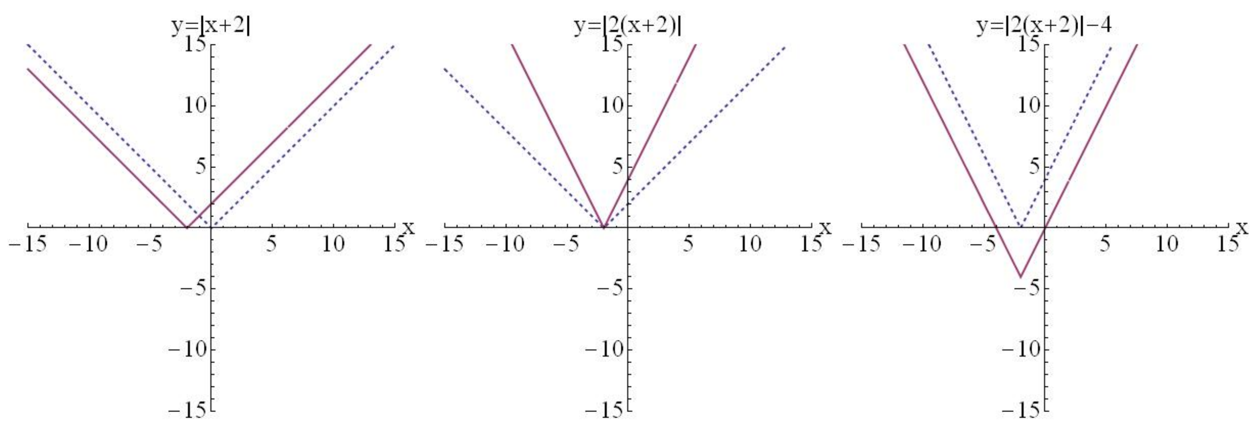
Example $y = 9x^2$ can be either a vertical stretch or horizontal shrink of the graph $y = x^2$.

If $y = f(x) = x^2$, then $y = 9f(x) = 9x^2$ represents a function which is found from $y = f(x)$ by stretching the graph vertically by a factor of 9.
If $y = f(x) = x^2$, then $y = f(3x) = (3x)^2 = 9x^2$ represents a function which is found from $y = f(x)$ by shrinking the graph horizontally by a factor of 3.



Example Find the formula for the graph obtained from the graph of $y = |x|$ using a shift left two units, then a vertical stretch by a factor of 2, and finally a shift down of 4 units.

Here is the algebraic representation:
Initial function: $y = |x|$.
Shift left 2 units: $y = |x + 2|$.
Vertical stretch by a factor of 2: $y = 2|x + 2|$.
Shift down 4 units: $y = 2|x + 2| - 4$.
Here is the graphical representation:

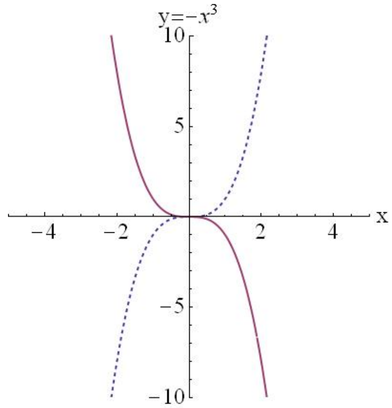


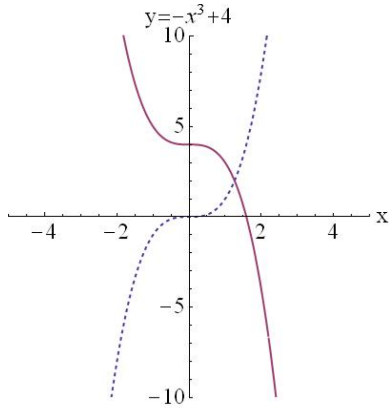
Example Find an algebraic expression for the function that is found from $y = f(x)$ by shifting to the right by 5 units, then compressing horizontally by 3 units, and then reflecting about the y-axis. Show each step in constructing the final function.

Original Function: $y = f(x)$.
Shift to the right 5 units: $y = f(x - 5)$.
Compress horizontally by 3 units: $y = f(3(x - 5))$.
Reflecting about the y-axis: $y = f(3(-x - 5))$.

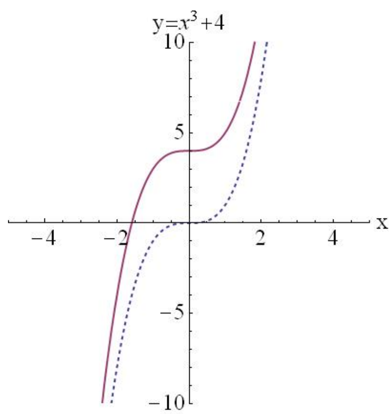
Example Show graphically that the transformations reflecting about the x axis and shifting up 4 units do not commute for the function $y = x^3$.

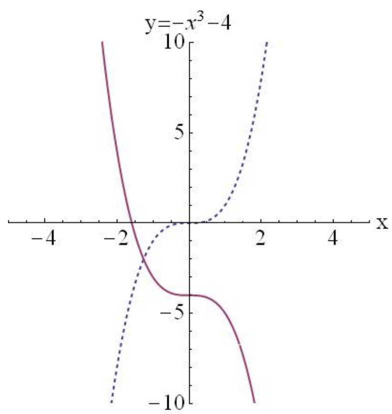
Reflect about the x-axis, then shift up 4 units:
Original Function: $y = f(x) = x^3$.
Reflecting about the x-axis: $y = -f(x) = -x^3$.
Shift up 4 units: $y = -f(x) + 4 = -x^3 + 4$.





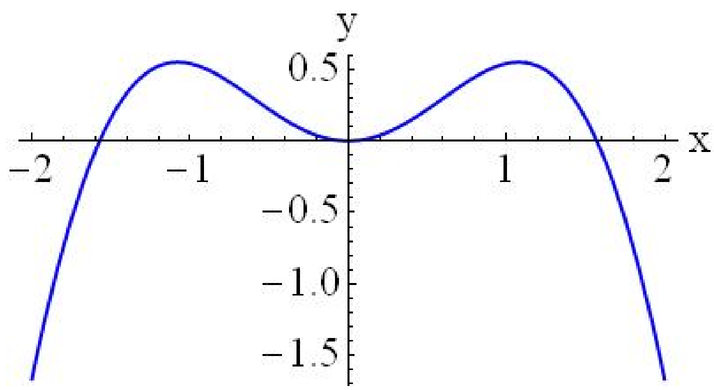
Shift up 4 units, then reflect about the x-axis:
Original Function: $y = f(x) = x^3$.
Shift up 4 units: $y = f(x) + 4 = x^3 + 4$.
Reflecting about the x-axis: $y = -(f(x) + 4) = -x^3 - 4$.



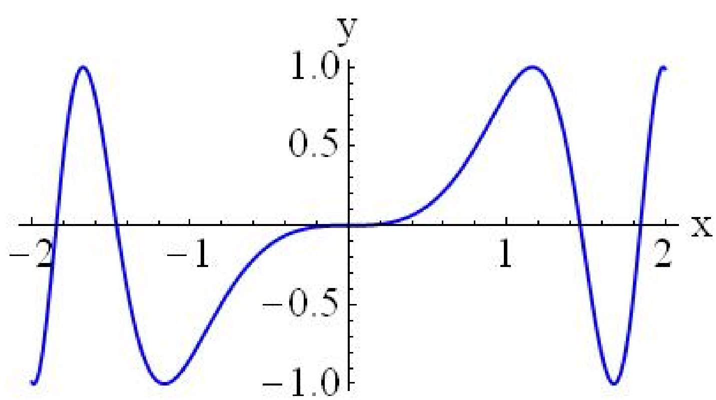


Symmetry

Even functions satisfy $f(-x) = f(x)$. Geometrically this means the function is symmetric about the y-axis.



Odd functions satisfy $f(-x) = -f(x)$. Geometrically this means the function is symmetric if we rotate 180 degrees about the origin.



NOTE: A function can be either even, or odd, or neither!