Concepts: Sketching Quadratic Functions, The Vertex Form of a Quadratic Function, Solving Quadratic Inequalities with Sign Charts.

Creating Good Sketches

When you are sketching without using graph paper (which we often do), here are some important things to do:

- Label your axes, x axis to the right (in the direction of increasing x) and y axis to the top (in the direction of increasing y).
- Include arrows on the ends of your curve if the line continues forever.
- Include the equation of the curve somewhere on the graph beside the curve.
- Explicitly label the points of interest on the curve as (x, y) ordered pairs.
- Exact numbers should be used whenever possible, but you may want to convert to a decimal to determine if a number is positive or negative (useful for determining which side of y-axis x-intercepts should be on in your sketch).
- Any annotations you make on the graph should be large and neatly labeled so it is easy to read.
- Make the entire graph large enough to easily read, and redraw it if necessary.
- Whatever work was done do help you create the sketch is NOT scratchwork, it is an important part of your solution. It must be neat, and organized in a manner that easily guides the reader through the creation of your sketch.

For quadratics,

- The vertex and x-intercept (if there are any) should be labelled.
- You may find it useful to use headings for computation of zeroes or vertex to help organize your work.
- Since there are two ways to sketch a quadratic (converting to vertex form, or looking at zeros, opens up/down and end-behaviour), tell the reader right at the start which way you are going to sketch.

Sketching Quadratics by Converting to The Vertex Form

The form $f(x) = a(x - h)^2 + k$ is called the vertex form for a quadratic function. It is obtained from the standard form $f(x) = ax^2 + bx + c$ by completing the square.

The vertex of the parabola is (h, k).
The axis of symmetry is $x = h$.
If $a > 0$, the parabola opens up, if $a < 0$ the parabola opens down.

You can also sketch a parabola by writing it in the vertex form and identifying the vertex, determining if it opens up/down, and finding any x-intercepts.

The domain of every quadratic is the set of real numbers, $\mathbb{R}$.

The range depends on whether the quadratic opens up or down:
Opens up, range $y \in [h, \infty)$; opens down, range $y \in (-\infty, h]$.

Sketching Quadratics by Computing Zeros, End Behaviour, and Opening Up/Down

For a quadratic function $y = f(x) = ax^2 + bx + c$, we can create a sketch by determining four things:

1. **x-intercepts:** determine these by using the quadratic formula $x_{\text{intercept}} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

   NOTE: If $b^2 - 4ac < 0$ then there are no x-intercepts (they are not real numbers).

2. **Vertex:** $(x_{\text{Vertex}}, y_{\text{Vertex}}) = \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

   $$x_{\text{Vertex}} = -\frac{b}{2a}$$
   $$y_{\text{Vertex}} = f\left(-\frac{b}{2a}\right)$$

   How to remember this: The x-coordinate of the vertex will be right in the middle of the two x-intercepts, even if the x-intercepts are not real numbers!

   $$\begin{aligned} x_{\text{Vertx}} &= \frac{1}{2}\left(\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}\right) \\ &= \frac{1}{2}\left(\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}\right) \\ &= \frac{1}{2}\left(\frac{-2b}{2a}\right) \\ &= \frac{-b}{2a} \end{aligned}$$

3. The function opens up if $a > 0$, and opens down if $a < 0$.

4. **y-intercept:** evaluate at $x = 0$, so figure out what $f(0)$ is. This can be added at the end, and is not needed to get the sketch.

Example Sketch the parabola $y = 5x^2 + 4x - 12$. Label the vertex, y-intercept, and any x-intercepts on your sketch.
To get the sketch of a quadratic, we should do four things:

1. Determine if it opens up or down,
2. Determine the vertex,
3. Determine any x-intercepts (if they exist),
4. Determine the y-intercept.

A quadratic opens up if $a > 0$, and opens down if $a < 0$. Since $a = 5$ in this case, this quadratic opens up.
To get the vertex, identity $a = 5, b = 4$, and $c = -12$. Then the vertex is located at:

$$x = \frac{-b}{2a} = \frac{-4}{2(5)} = -\frac{2}{5}$$
$$y = f\left(\frac{-b}{2a}\right) = f\left(-\frac{2}{5}\right) = 5\left(-\frac{2}{5}\right)^2 + 4\left(-\frac{2}{5}\right) - 12 = -\frac{64}{5}$$

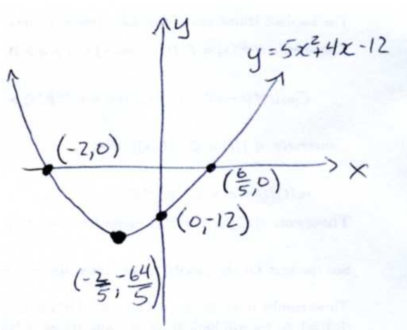
The vertex is at $\left(-\frac{2}{5}, -\frac{64}{5}\right)$.
To get the x-intercepts, use the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(5)(-12)}}{2(5)} \\ &= \frac{-4 \pm \sqrt{256}}{10} \\ &= \frac{-4 \pm 16}{10} \\ &= \frac{-4+16}{10} \text{ or } \frac{-4-16}{10} \\ &= \frac{12}{10} \text{ or } \frac{-20}{10} \\ &= \frac{6}{5} \text{ or } -2 \end{aligned}$$

To get the y-intercept, evaluate $f(0)$:

$$y = f(0) = 5(0)^2 + 4(0) - 12 = -12$$

You can now put this all together to get the sketch:



Sign Charts to Solve Inequalities

A sign chart simply lists the sign of the function you are interested in along the x-axis. It does not provide as much information as a sketch, but it provides enough to solve inequalities.

Constructing a sign chart for a quadratic relies on knowing the *end behaviour* of the quadratic, and the x-intercepts. We will see more complicated sign charts for polynomials and rational functions later.

Note: The text describes sign charts slightly differently, in that they use a test-point method to determine the sign of the quadratic in each interval once the x-intercepts are known. Therefore, they do not need to know the end behaviour when they construct their sign chart. The technique I have outlined below and the one the text uses are both common techniques to generate sign charts and solve inequalities.

Example Determine the solution to $2x^2 - 4x - 9 < 0$ using a sign chart. Write the solution using interval notation.

First, we must determine if there are any x-intercepts for the quadratic, which we can do using the quadratic formula.

$$\begin{aligned} 2x^2 - 4x - 9 = 0 \\ x &= \frac{4 \pm \sqrt{16 - 4(2)(-9)}}{4} \\ &= \frac{4 \pm \sqrt{88}}{4} \\ &= \frac{2 \pm \sqrt{22}}{2} \end{aligned}$$

so there are distinct real-valued x-intercepts. This is important, since this tells us the quadratic changes sign at $x = \frac{2 \pm \sqrt{22}}{2}$ (if there was only one real root, or the roots were complex-valued, the quadratic would not change sign).

Now, the quadratic opens up since $a = 2 > 0$.

A quadratic that opens up must at some point be positive for very large x; we can be more precise and write this as

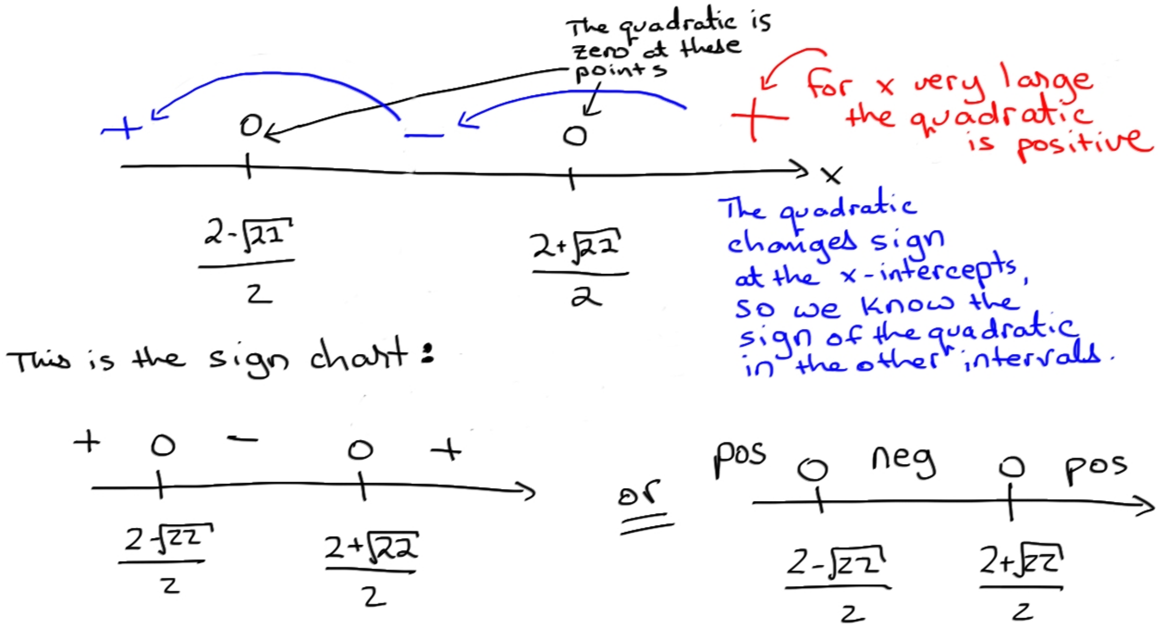
$$\lim_{x \to \infty} (2x^2 - 4x - 9) = \infty$$

and

$$\lim_{x \to -\infty} (2x^2 - 4x - 9) = \infty$$

Precise understanding of the limit notation will happen in calculus, but it is nice to see it here.

Putting all this information together, we can construct a *sign chart*:



And from the sign chart, we read of where the quadratic is less than zero: $x \in \left(\frac{2-\sqrt{22}}{2}, \frac{2+\sqrt{22}}{2}\right)$.