Concepts: Higher degree polynomials, Remainder Theorem, Factor Theorem, Rational Zero Theorem, Fundamental Theorem of Algebra.

Polynomials of degree 1 and 2 are so important because we can do so much with them; get x-intercepts, sketch graphs quickly, and basically understand them fully.

Higher degree polynomials are more difficult to describe. There is no beautiful technique like completing the square (which gave us the quadratic formula) we can use easily that will work in every single case–actually, there are techniques for polynomials of degree 3 and 4, but the techniques are vastly more complicated than completing the square and generally only implemented by a computer!

In many cases, however, we can still sketch the graphs of higher degree polynomials by hand, and describe general behaviour. Before we can ge to sketching, we have to investigate techniques that can be used to factor higher degree polynomials.

Vocabulary

The standard form for writing a polynomial of degree n is $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

Each monomial ($a_n x^n$, $a_{n-1}x^{n-1}$, etc.) is a *term* of the polynomial.
The a_i for $i = 0, 1, 2, 3, \ldots, n$ are the *coefficients* of the polynomial.
The term $a_n x^n$ is the *leading term* of the polynomial.

Recall: Factoring Rules for Quadratics and Sum/Difference of Cubes

$$
\begin{aligned}
x^2 + 2xy + y^2 &= (x+y)^2 \\
x^2 - y^2 &= (x+y)(x-y) \\
x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\
x^3 - y^3 &= (x-y)(x^2 + xy + y^2)
\end{aligned}
$$

If you cannot factor a quadratic by inspection, you can use the quadratic formula to factor.

Long Division of Polynomials is also useful for factoring polynomials. I do not use synthetic division, although it is discussed in detail in this section and you are of course free to use it if you like.

There are some important theorems to help us factor polynomials, and our goal here is to understand what they tell us and how to use them to assist in factoring polynomials.

The Remainder Theorem: If a polynomial f is divided by $x - c$, then the remainder is $f(c)$. This means that if $f(c) = 0$, then $x = c$ is a root of f.

The Remainder Theorem can therefore help us make educated guesses to find factors of polynomials.

The Factor Theorem: The number c is a zero of the polynomial $y = P(x)$ if and only if $x - c$ is a factor of the polynomial $P(x)$.

The Factor Theorem tell us how zeros and factors are related.

Example Try to guess a factor of the polynomial $f(x) = x^3 - 2x^2 - 5x + 6$. Use your guess to factor $f(x)$ completely.
The Remainder Theorem tells us how to make educated guesses by telling us to look for c such that $f(c) = 0$.
We see that $f(1) = 1 - 2 - 5 + 6 = 0$.
So The Factor Theorem tells us since $c = 1$ is a zero, therefore $x - 1$ is a factor.
We then proceed to factor $x - 1$ out of $f(x)$ using polynomial long division.

$$\begin{array}{r l l}
 & x^2 - x - 6 & \\
x-1 \,\big) & x^3 - 2x^2 - 5x + 6 & \\
 & \underline{x^3 - x^2} & \text{subtract} \\
 & -x^2 - 5x + 6 & \\
 & \underline{-x^2 + x} & \text{subtract} \\
 & -6x + 6 & \\
 & \underline{-6x + 6} & \text{subtract} \\
 & 0 & \text{remainder is zero, as expected.}
\end{array}$$

This shows $f(x) = (x - 1)(x^2 - x - 6)$, and we can factor the quadratic by inspection, so

$$f(x) = (x - 1)(x - 3)(x + 2).$$

The zeros of $f(x)$ are $x = 1, 3, -2$.
The factors of $f(x)$ are $(x - 1)$, $(x - 3)$ and $(x + 2)$.

A natural question to ask, is does every polynomial have a zero? The Fundamental Theorem of Algebra answers this question.

The Fundamental Theorem of Algebra: If $y = P(x)$ is a polynomial function of positive degree, then $y = P(x)$ has at least one zero in the set of complex numbers.

Notice that the polynomial is guaranteed to have at least one complex zero, but not necessarily a real-values zero. This makes sense, since for example $f(x) = x^2 + 4$ has no real-valued zeros, but does have two complex-valued zeros, $x = \pm 2i$. This is an important fact moving forward, since we might be tempted to think of "zeros" and "x-intercepts" as the same thing, but this example shows they are not. The polynomial $f(x) = x^2 + 4$ has no x-intercepts, but does have two complex-valued zeros.

In practice, we would like to have a better technique than simply making educated guesses for zeros of polynomials. The Rational Zero Theorem give us even more insight to the zeros of polynomials.

Rational Zeros Theorem Suppose f is a polynomial of degree $n \geq 1$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

with every coefficient an integer and $a_0 \neq 0$. If $x = p/q$ is a rational zero of f, where p and q have no common integer factors other than 1, then

1) p is an integer factor of the the constant coefficient a_0.
2) q is an integer factor of the leading coefficient a_n.

Here is an example of how we can use The Rational Zero Theorem to improve on our educated guesses.

Example Find all of the real zeros of the function $f(x) = 2x^3 - 3x^2 - 4x + 6$.

Since the coefficients are all integers, we will use the Rational Zero Theorem to get us started.
Factors of $a_0 = 6$: $\pm 1, \pm 2, \pm 3, \pm 6$.
Factors of $a_4 = 2$: $\pm 1, \pm 2$.
Potential rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$.

Evaluate $f(x)$ at these potential rational zeros until you find a zero.

$$\begin{aligned} f(1) &= 2(1)^3 - 3(1)^2 - 4(1) + 6 = +1 \neq 0 \\ f(1/2) &= 2(1/2)^3 - 3(1/2)^2 - 4(1/2) + 6 = 7/2 \neq 0 \\ f(3/2) &= 2(3/2)^3 - 3(3/2)^2 - 4(3/2) + 6 = 0 \end{aligned}$$

Therefore, $x = 3/2$ is a root. Let's factor it out using long division.

$$\begin{array}{r|l} & 2x^2 - 4 \\ \hline x - \frac{3}{2} & 2x^3 - 3x^2 - 4x + 6 \\ & 2x^3 - 3x^2 \\ \cline{2-2} & \qquad\quad -4x + 6 \\ & \qquad\quad -4x + 6 \\ \cline{2-2} & \qquad\qquad\quad 0 \end{array}$$

$$f(x) = 2x^3 - 3x^2 - 4x + 6 = \left(x - \frac{3}{2}\right)(2x^2 - 4) = (2x - 3)(x^2 - 2) = (2x - 3)(x + \sqrt{2})(x - \sqrt{2})$$

There are three real roots, one rational ($x = 3/2$) and two irrational ($x = \pm\sqrt{2}$). Each root has multiplicity one.

Note: This question could be asked on a test in the following manner:
Find all of the real zeros of the function $f(x) = 2x^3 - 3x^2 - 4x + 6$ given $x = 3/2$ is a zero of f.

Example Find the zeros of the polynomial $f(x) = 5x^3 - 5x^2 - 10x$.

We do this by factoring:

$$\begin{aligned} f(x) &= 5x^3 - 5x^2 - 10x \\ &= 5x(x^2 - x - 2) \\ &= 5x(x + 1)(x - 2) \end{aligned}$$

The zeros are $x = 0, -1, 2$. Done!

If we can't factor a quadratic by inspection, we can use the quadratic formula to find the roots when we get to a quadratic.