**Concepts:** Multiplicity, $n$-Root Theorem, Conjugate Pairs Theorem, root/zero/$x$-intercept, Descartes's Rule of Signs

**Definition (multiplicity):** If the polynomial $f$ has $(x-c)^m$ as a factor but not $(x-c)^{m+1}$, then $c$ is a *zero of f of multiplicity m*.

**Example** Find all the roots of $f(x) = \frac{9}{4}x^2 - 21x + 49$.
Seeing that the $\frac{9}{4} = (\frac{3}{2})^2$ and $49 = 7^2$, we might guess this factors as a perfect square, $f(x) = (\frac{3}{2}x - 7)^2$. Let's check if our guess is correct:

$$\begin{aligned}(\frac{3}{2}x-7)^2 &= (\frac{3}{2}x-7)(\frac{3}{2}x-7)\\ &= \frac{9}{4}x^2 - \frac{21}{2}x - \frac{21}{2}x + 49\\ &= \frac{9}{4}x^2 - 21x + 49\\ &= f(x)\end{aligned}$$

So since $f(x) = (\frac{3}{2}x - 7)^2$, $f$ has a root of $c = 14/3$ of multiplicity 2.

**$n$-Root Theorem:** If $P(x) = 0$ is a polynomial equation with real or complex coefficients and positive degree $n$, then (including multiplicity) $P(x) = 0$ has $n$ roots.

**Conjugate Pairs Theorem:** If $P(x) = 0$ is a polynomial equation with real coefficients and the complex number $c = a + bi$ is a root, then the complex conjugate $\bar{c} = a - bi$ is also a root.

These theorems tell us that any polynomial of degree $n$ with real coefficients can be written as

$$P(x) = (x-c_1)(x-c_2)(x-c_3)\ldots(x-c_n) = \sum_{i=1}^{n}(x-c_i),$$

where some of the $c_i \in \mathbb{C}$ may be repeated, and complex roots appear in complex conjugate pairs.

Although we won't be looking at sketching polynomials by hand until later, you may be using a calculator to sketch polynomials to help you visualize examples, and there are some things to note about the sketches of polynomials.

When we look at a sketch of a polynomial of degree $n$, we may not see $n$ $x$-intercepts due to two things:

- some of the zeros may be real, but have multiplicity greater than one, and
- some of the zeros may be complex.

**How Multiplicity of zero $c \in \mathbb{R}$ Affects the Behaviour of $f(x)$**

- If $c \in \mathbb{R}$ is a zero of the polynomial $f$ with **odd multiplicity**, then the graph of $f$ **crosses the $x$ axis at** $x = c$. This is because the function $f$ will change sign at $x = c$.
- If $c \in \mathbb{R}$ is a zero of the polynomial $f$ with **even multiplicity**, then the graph of $f$ **does not cross the $x$ axis at** $x = c$, but does touch the $x$ axis at $x = c$. This is because the function $f$ will not change sign at $x = c$.
- If the multiplicity is greater than or equal to 2, the graph will be horizontal where it touches the $x$-axis.

**Equivalent Statements for Polynomial Functions**

All these statements are equivalent if $c \in \mathbb{R}$. If one is true, all the others are true as well.

1. $x = c$ is a root of the equation $f(x) = 0$.
2. $c$ is a zero of the function $f$.
3. $c$ is an $x$-intercept of the graph of $y = f(x)$.
4. $x - c$ is a factor of $f(x)$.

**Note:** If $c$ is a complex number, it can be a root but not an $x$-intercept. If $c \in \mathbb{C}$, Statements 1, 2, and 4 are all equivalent.

**Example** Find the zeros with multiplicity for the polynomial $f(x) = x(3x-5)^4(2+x)^3$. What do you know about the behaviour of $f$ near the zeros from this?

- zero at $x = 0$ has multiplicity 1 (odd) so $f$ changes sign at $x = 0$,
- zero at $x = 5/3$ has multiplicity 4 (even) so $f$ does not change sign at $x = 5/3$,
- zero at $x = -2$ has multiplicity 3 (odd) so $f$ changes sign at $x = -2$, and since multiplicity was greater than 2, $f$ will be horizontal at $x = -2$.

**Definition: Variation of Signs** For a polynomial written in descending order, we say a *variation of signs* occurs when the sign of consecutive terms changes. For example

$P(x) = +2x^3 - 3x^2 + 5x - 6$ 3 variations of sign

$P(x) = -2x^4 - 7x^3 + 3x^2 + x - 1$ 2 variations of sign

To use Descartes's Rule of Signs, we also need to check the variation of sign of $P(-x)$ once it has been simplified.

$P(x) = 2x^3 - 3x^2 + 5x - 6$ has 3 variations of sign.
$P(-x) = 2(-x)^3 - 3(-x)^2 + 5(-x) - 6$
$= -2x^3 - 3x - 5x - 6$ has no variations of sign.

$P(x) = -2x^4 - 7x^3 + 3x^2 + x - 1$ has 2 variations of sign.
$P(-x) = -2(-x)^4 - 7(-x)^3 + 3(-x)^2 + (-x) - 1$
$= -2x^4 + 7x^3 + 3x^2 - x - 1$ has 2 variations of sign.

## Descartes's Rule of Signs

Suppose $P(x) = 0$ is a polynomial equation with real coefficients with terms written in descending order.

- The number of positive real roots of the equation is either equal to the number of variations of sign of $P(x)$ or less than that by an even number.
- The number of negative real roots of the equation is either equal to the number of variations of sign of $P(-x)$ or less than that by an even number.

Descartes's Rule allows us to say something about the roots of $P(x) = 0$ without actually solving the equation. This can be important since solving the equation may be quite difficult!

**Example:** Discuss the possibilities for the roots to $2x^3 - 3x^2 + 5x - 6 = 0$ using Decsartes's Rule of Signs.
$P(x) = 2x^3 - 3x^2 + 5x - 6$ has 3 variations of signs, so the number of positive real roots is either 3 or 1.

$P(-x) = -2x^3 - 3x^2 - 5x - 6$ has no variations of signs, so the number of negative real roots is definitely 0.

Since the equation is third degree, there must be three roots, and the possibilities are

- 3 positive real roots, or
- 1 positive real root, 2 complex roots (complex conjugate pairs).

**Example:** Discuss the possibilities for the roots to $-2x^4 - 7x^3 + 3x^2 + x - 1 = 0$ using Decsartes's Rule of Signs.
$P(x) = -2x^4 - 7x^3 + 3x^2 + x - 1$ has 2 variations of signs, so the number of positive real roots is either 2 or 0.

$P(-x) = -2x^4 + 7x^3 + 3x^2 - x - 1$ has 2 variations of signs, so the number of negative real roots is either 2 or 0.

Since the equation is fourth degree, there must be four roots, and the possibilities are

- 2 positive real roots and 2 negative real roots,
- 2 positive real roots and 2 complex roots (complex conjugate pairs),
- 2 negative real roots and 2 complex roots (complex conjugate pairs),
- 4 complex roots, appearing in 2 sets of complex conjugate pairs.

## Bounds on Roots

I am not having you do the Bounds on Roots section, since it never comes up in calculus. It can be useful, but it is not as critical as the other aspects.