Concepts: Factoring Higher Degree Equations, Equations Involving Square Roots, Equations with Rational Exponents, Equations of Quadratic Type, Equations Involving Absolute Values.

This Section collects a variety of different types of equations and shows how what we have learned can be used to solve them. All these rely on the zero factor property: The equation $AB = 0$ is equivalent to the compound statement $A = 0$ or $B = 0$.

Factoring Higher Degree Equations

Sometimes you can factor by grouping, and sometimes you have to use the ideas from Section 3.2. This section focusses on simple factoring rather than what was done in Section 3.2, but realize the techniques from 3.2 may have to be applied in some cases.

It is critical that you solve by factoring and not canceling or you can miss some solutions!

Example Solve $x^5 = 27x^2$.

$$x^5 = 27x^2$$
$$x^5 - 27x^2 = 0$$
$$x^2(x^3 - 27) = 0$$
$$x^2(x^3 - 3^3)\& = 0 \text{ (difference of cubes)}$$
$$x^2(x-3)(x^2+3x+9)\& = 0$$

Now use the zero factor property:

$$x^2 = 0 \text{ or } x - 3 = 0 \text{ or } x^2 + 3x + 9 = 0$$

The first two give solutions $x = 0$ (multiplicity 2), and $x = 3$. The third requires the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{9 - 36}}{2} \\ &= \frac{-3 \pm \sqrt{-27}}{2} \\ &= \frac{-3 \pm \sqrt{27}i}{2} \end{aligned}$$

So there are four solutions, two real values and two complex conjugates:

$$x = 0 \text{ (multiplicity 2)}, 3, \frac{-3 \pm \sqrt{27}i}{2}.$$

Equations Involving Square Roots

To solve equations with square roots, you first isolate a single square root on one side of the equation, and then square both sides of the equation. You may have to do this more than once to eliminate all the square roots. When the square roots are gone, you will be left with a polynomial that can be solved by factoring.

Note: These algebraic manipulations (squaring both sides of the equation) may introduce extraneous solutions, so you must check that the solutions you find satisfy the original equation to eliminate extraneous solutions.

Note: Now is a good time to remind yourself that $\sqrt{A^2+B^2} \neq A+B$ and $(\sqrt{A}+\sqrt{B})^2 \neq A+B$. What is true is that $(\sqrt{A})^2 = A$ and $(A+B)^2 = A^2+2AB+B^2$.

Example Solve $\sqrt{y+10}-\sqrt{y-2}=2$.

$$\begin{aligned}
\sqrt{y+10}-\sqrt{y-2} &= 2 \\
\sqrt{y+10} &= 2+\sqrt{y-2} \\
(\sqrt{y+10})^2 &= (2+\sqrt{y-2})^2 \\
y+10 &= 4+4\sqrt{y-2}+(\sqrt{y-2})^2 \\
y+10 &= 4+4\sqrt{y-2}+y-2 \\
2 &= \sqrt{y-2} \\
2^2 &= (\sqrt{y-2})^2 \\
4 &= y-2 \\
y &= 6
\end{aligned}$$

Check: $\sqrt{6+10}-\sqrt{6-2} = 4-2 = 2$ so $y=6$ is a solution.

Example Solve $\dfrac{1}{z} = \dfrac{3}{\sqrt{4z+1}}$.

There are a variety of intermediate steps possible that will lead you to the correct solution.

$$\begin{aligned}
\frac{1}{z} &= \frac{3}{\sqrt{4z+1}} \\
\sqrt{4z+1} &= 3z \\
(\sqrt{4z+1})^2 &= (3z)^2 \\
4z+1 &= 9z^2 \\
9z^2-4z-1 &= 0
\end{aligned}$$

Now use the quadratic formula:

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{16+36}}{2(9)} \\ &= \frac{4 \pm \sqrt{52}}{18} \\ &= \frac{4 \pm \sqrt{4 \cdot 13}}{18} \\ &= \frac{4 \pm 2\sqrt{13}}{18} \\ &= \frac{\cancel{2}(2 \pm \sqrt{13})}{\cancel{2} \cdot 9} \\ &= \frac{2 \pm \sqrt{13}}{9} \end{aligned}$$

Now we need to check. First, the solution $\frac{2+\sqrt{13}}{9}$:

$$\begin{aligned} \frac{3}{\sqrt{4z+1}} &= \frac{3}{\sqrt{4(\frac{2+\sqrt{13}}{9})+1}} \\ &= \frac{3}{\sqrt{\frac{8+4\sqrt{13}}{9} + \frac{9}{9}}} \\ &= \frac{3}{\sqrt{\frac{17+4\sqrt{13}}{9}}} \\ &= \frac{3 \cdot \sqrt{9}}{\sqrt{17+4\sqrt{13}}} \\ &= \frac{9}{\sqrt{17+4\sqrt{13}}} \\ \frac{1}{z} &= \frac{1}{\frac{2+\sqrt{13}}{9}} \\ &= \frac{9}{2+\sqrt{13}} \end{aligned}$$

It might look like these are two different numbers, but if we convert to decimals, we will see they are the same. Since the denominators are what are different, I will (rather than converting to decimals) show the denominators are the same number. Since both denominators are positive, if I square both sides and get the same number the denominators are equal.

$$\begin{aligned} (\sqrt{17+4\sqrt{13}})^2 &= 17 + 4\sqrt{13} \\ (2+\sqrt{13})^2 &= 4 + 4\sqrt{13} + 13 = 17 + 4\sqrt{13} \end{aligned}$$

So, $z = \frac{2+\sqrt{13}}{9}$ is a solution.

Now, the solution $\frac{2-\sqrt{13}}{9}$:

$$\begin{aligned}
\frac{3}{\sqrt{4z+1}} &= \frac{3}{\sqrt{4(\frac{2-\sqrt{13}}{9})+1}} \\
&= \frac{3}{\sqrt{\frac{8-4\sqrt{13}}{9}+\frac{9}{9}}} \\
&= \frac{3}{\sqrt{\frac{17-4\sqrt{13}}{9}}} \\
&= \frac{3\cdot\sqrt{9}}{\sqrt{17-4\sqrt{13}}} \\
&= \frac{9}{\sqrt{17-4\sqrt{13}}} \\
\frac{1}{z} &= \frac{1}{\frac{2-\sqrt{13}}{9}} \\
&= \frac{9}{2-\sqrt{13}}
\end{aligned}$$

Now, the problem here is subtle at this point, but obvious if you convert to a decimal. You do not need to convert to a decimal to see this, it just makes it easier to see.

$$\begin{aligned}
\sqrt{17-4\sqrt{13}} &= 1.60555 > 0 \\
2-\sqrt{13} &= -1.60555 < 0
\end{aligned}$$

So this is actually not a solution.

Equations with Rational Exponents

These types are similar to the square root equations (since $\sqrt{x} = x^{1/2}$), so we isolate a piece with a rational exponent and then raise both sides of the equation to an appropriate power to get rid of the rational exponent.

Note: Here we need to know $x^{m/n} = (\sqrt[n]{x})^m$, and the rules of exponents.

These types can actually get quite difficult if the rational exponent is anything except $m/2$ or $m/3$, since to remove you would have to raise both sides of the equation to n and if $n = 4$ that could leave you with a 4th degree polynomial to solve! For this reason, most often you will see this technique used when you have square roots or cube roots, even though it applies in general and will work for more complicated equations.

Advice: Since you may have to simplify by taking roots, do not expand out powers of numbers until the end. The second example below shows what I mean.

Example Solve $(2-s)^{-1/2} = 1/3$.

$$\begin{aligned}
((2-s)^{-1/2})^{-2} &= (1/3)^{-2} \\
2-s &= (3)^2 \\
2-s &= 9 \\
s &= -7
\end{aligned}$$

Check: $(2+7)^{-1/2} = (9)^{-1/2} = 1/3$ so $s = -7$ is a solution.

Example Solve $w^{-4/3} = 16$.

$$
\begin{aligned}
w^{-4/3} &= 16 \\
(w^{-4/3})^{-3/4} &= (16)^{-3/4} \\
w &= (16)^{-3/4} \\
&= ((16)^{1/4})^{-3} \\
&= ((2^4)^{1/4})^{-3} \\
&= (\pm 2)^{-3} \text{ (even root, so we get plus or minus)} \\
&= (\pm\frac{1}{2})^3 \\
&= \pm\frac{1}{8}
\end{aligned}
$$

Both these satisfy the original equation.

Equations of Quadratic Type

These are equations that you can use the quadratic form on, but the x is more complicated $x = u$: $a[u]^2+b[u]+c = 0$. Typically, $u = cx + d$, $u = x^2$, $u = cx^2 + dx$, or $u = x^{m/n}$.

First, solve the quadratic in u using the quadratic formula, and then put in what u is in terms of x and solve these equations for x.

You may have to use the quadratic formula twice in some cases!

Example: Solve $(v^2 - 4v)^2 - 17(v^2 - 4v) + 60 = 0$.

Here, $u = v^2 - 4v$. I make that substitution and then solve for u:

$$u^2 - 17u + 60 = 0$$

$$
\begin{aligned}
u &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{17 \pm \sqrt{(-17)^2 - 4(1)(60)}}{2(1)} \\
&= \frac{17 \pm \sqrt{49}}{2} \\
&= \frac{17 \pm 7}{2} \\
&= \frac{17+7}{2} \text{ or } \frac{17-7}{2} \\
&= 12 \text{ or } 5
\end{aligned}
$$

Now we solve two equations for v:

$$\begin{aligned}
v^2 - 4v &= 12 \\
v^2 - 4v - 12 &= 0 \\
v &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-12)}}{2(1)} \\
&= \frac{4 \pm \sqrt{64}}{2} \\
&= \frac{4 \pm 8}{2} \\
&= \frac{4+8}{2} \text{ or } \frac{4-8}{2} \\
&= 6 \text{ or } -2
\end{aligned}$$

$$\begin{aligned}
v^2 - 4v &= 5 \\
v^2 - 4v - 5 &= 0 \\
v &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)} \\
&= \frac{4 \pm \sqrt{36}}{2} \\
&= \frac{4 \pm 6}{2} \\
&= \frac{4+6}{2} \text{ or } \frac{4-6}{2} \\
&= 5 \text{ or } -1
\end{aligned}$$

The solutions are $v = -2, -1, 5, 6$. You can check that these satisfy the original equation.

Example: Solve $x^{2/3} + 10 = 7x^{1/3}$.

Identify this as quadratic in $u = x^{1/3}$, so rewrite it is $u^2 + 10 = 7u$.

$$\begin{aligned} u^2 + 10 &= 7u \\ u^2 - 7u + 10 &= 0 \\ u &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{7 \pm \sqrt{(-7)^2 - 4(1)(10)}}{2(1)} \\ &= \frac{7 \pm \sqrt{9}}{2} \\ &= \frac{7 \pm 3}{2} \\ &= \frac{7+3}{2} \text{ or } \frac{7-3}{2} \\ &= 5 \text{ or } 2 \end{aligned}$$

Now we solve two equations for x:

$$\begin{aligned} x^{1/3} &= 5 \\ x &= 5^3 = 125 \end{aligned}$$

$$\begin{aligned} x^{1/3} &= 2 \\ x &= 2^3 = 8 \end{aligned}$$

The two solutions are $x = 8, 125$. You can check these satisfy the original equation.

Equations Involving Absolute Values

The techniques to solve absolute value equations were developed in Section 1.1. We use the same techniques on more complicated problems here.

Note: You need to know $|A| = B$ is equivalent to the compound statement $A = -B$ or $A = B$.

Example Solve $|x^2 - 2x| = |3x - 6|$.

This is equivalent to $x^2 - 2x = 3x - 6$ or $x^2 - 2x = -(3x - 6)$. Solve each in turn.

$$\begin{aligned}
x^2 - 2x &= 3x - 6 \\
x^2 - 5x + 6 &= 0 \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{5 \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} \\
&= \frac{5 \pm \sqrt{1}}{2} \\
&= \frac{5 \pm 1}{2} \\
&= \frac{5 + 1}{2} \text{ or } \frac{5 - 1}{2} \\
&= 3 \text{ or } 2
\end{aligned}$$

$$\begin{aligned}
x^2 - 2x &= -(3x - 6) \\
x^2 + x - 6 &= 0 \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-6)}}{2(1)} \\
&= \frac{-1 \pm \sqrt{25}}{2} \\
&= \frac{-1 \pm 5}{2} \\
&= \frac{-1 + 5}{2} \text{ or } \frac{-1 - 5}{2} \\
&= 2 \text{ or } -3
\end{aligned}$$

The three solutions are $x = -3, 2, 3$. You can check these satisfy the original equation.