Concepts: Intermediate Value Theorem, Sketching Polynomials, Behaviour at the x-intercepts, Leading Coefficient Test (end behaviour), Solving Inequalities using Sign Charts.

In calculus, it will be important to be able to sketch polynomials by hand.

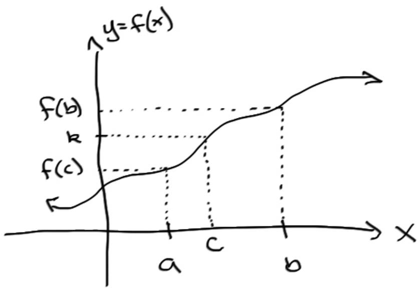
This section contains all the information you need to do so, reviewing some concepts we have seen before.

Intermediate Value Theorem

Sketching by hand rather than using a calculator since when sketching by hand you are able to identify all the important features of the function you are sketching, whereas on a calculator it is possible to miss important features. Some of the features we would be interested in you will learn about in calculus (most notably, location of extrema, and concavity).

The Intermediate Value Theorem is what allows a computer to sketch a function, and allows us to sketch a function by drawing a smooth line through the interesting features we have identified.

The Intermediate Value Theorem Suppose f is a polynomial function and $[a, b]$ is an interval for which $f(a) \neq f(b)$. If k is a number between $f(a)$ and $f(b)$, then there is a number c in the interval (a, b) such that $f(c) = k$.



Behaviour at the x-intercept

The behaviour at the x-intercepts is about whether or not the polynomial crosses the x-axis, which is related to the multiplicity of the zero that gives the x-intercept. We only have to worry about what is happening near the particular x-intercept we are examining.

Definition (multiplicity): If the polynomial f has $(x - c)^m$ as a factor but not $(x - c)^{m+1}$, then c is a *zero of f of multiplicity m*.

If $c \in \mathbb{R}$ is a zero of the polynomial f with odd multiplicity, then the graph of f crosses the x axis at $x = c$. This is because the function f will change sign at $x = c$.

If $c \in \mathbb{R}$ is a zero of the polynomial f with even multiplicity, then the graph of f does not cross the x axis at $x = c$. This is because the function f will not change sign at $x = c$.

If the multiplicity is greater than or equal to 2, the graph will be horizontal where it touches the x-axis.

Fundamental Truths for Polynomial Functions All these statements are equivalent if $c \in \mathbb{R}$. If one is true, all the others are true as well.

1) $x = c$ is a root of the equation $f(x) = 0$.
2) c is a zero of the function f.
3) c is an x-intercept of the graph of $y = f(x)$.
4) $x - c$ is a factor of $f(x)$.

Note: If c is a complex number, it can be a root but not an x-intercept. If $c \in \mathbb{C}$, Statements 1, 2, and 4 are all equivalent.

Leading Coefficient Test (End Behaviour)

For any function, we will be interested in the end behaviour, which we can more mathematically describe in terms of two limits:

$$\lim_{x\to\infty} f(x) \text{ “The limit as x approaches infinity of } f(x)\text{”}$$
$$\lim_{x\to-\infty} f(x) \text{ “The limit as x approaches minus infinity of } f(x)\text{”}$$

You will study these kinds of limits in much more detail in calculus (with more a more rigorous definition of limit). For the moment, I just want to continue to use the notation of limits where appropriate so you get used to seeing it before your calculus class.

If $f(x)$ is a polynomial, the end behaviour is entirely determined by the leading coefficient, since the leading term of the polynomial is dominant when $|x|$ is very large.

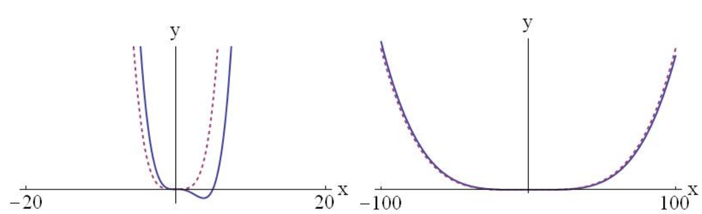
Note: The expression $|x|$ very large means x is very large OR x is very large negative, out beyond any of the x-intercepts the polynomial may have.

Example What is the end behaviour of the polynomial $f(x) = x^4 - 5x^3 + x - 1$?

We can determine the end behaviour by noting that the leading term is the dominant part of the polynomial for large values of x.

$$f(x) = x^4 - 5x^3 + x - 1 \sim x^4 \text{ for } x \text{ large, and } x \text{ large negative.}$$

This means the graph of $f(x)$ will approach the graph of its leading term for extreme values of x. Notice that if we zoom out to really see this (the graph on the right), it is impossible to see the interesting behaviour of $f(x)$ near the origin (left). This is one of the reasons why sketching by hand is so important.



In this example, we can say

$$\lim_{x\to\infty} (x^4 - 5x^3 + x - 1) = \infty$$
$$\lim_{x\to-\infty} (x^4 - 5x^3 + x - 1) = \infty$$

If the leading coefficient is positive for a polynomial of even degree, then:

$$\lim_{x\to\infty} f(x) = \infty$$
$$\lim_{x\to-\infty} f(x) = \infty$$

For example,

$$\lim_{x\to\infty} (\frac{17}{3}x^8 + x - 1) = \infty$$
$$\lim_{x\to-\infty} (\frac{17}{3}x^8 + x - 1) = \infty$$

If the leading coefficient is negative for a polynomial of even degree, then:

$$\lim_{x\to\infty} f(x) = -\infty$$
$$\lim_{x\to-\infty} f(x) = -\infty$$

For example,

$$\lim_{x\to\infty} (-\frac{17}{3}x^8 + x - 1) = -\infty$$
$$\lim_{x\to-\infty} (-\frac{17}{3}x^8 + x - 1) = -\infty$$

If the leading coefficient is positive for a polynomial of odd degree, then:

$$\lim_{x\to\infty} f(x) = \infty$$
$$\lim_{x\to-\infty} f(x) = -\infty$$

For example,

$$\lim_{x\to\infty} (\frac{17}{3}x^7 + x - 1) = \infty$$
$$\lim_{x\to-\infty} (\frac{17}{3}x^7 + x - 1) = -\infty$$

If the leading coefficient is negative for a polynomial of odd degree, then:

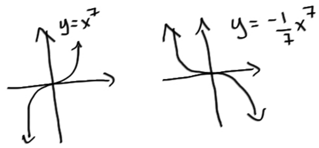
$$\lim_{x\to\infty} f(x) = -\infty$$
$$\lim_{x\to-\infty} f(x) = \infty$$

For example,

$$\lim_{x\to\infty} (-\frac{17}{3}x^7 + x - 1) = -\infty$$
$$\lim_{x\to-\infty} (-\frac{17}{3}x^7 + x - 1) = \infty$$

Advice: Work this out rather than memorizing by remembering the sketches of the leading coefficients.

Example What is the end behaviour of the polynomial $f(x) = -\frac{1}{7}x^7 + 789x^6 + 786x^5 + 9x^4 - x^3 - 66x$?



$$\lim_{x\to\infty}\left(-\frac{1}{7}x^7\right) = -\infty$$
$$\lim_{x\to-\infty}\left(-\frac{1}{7}x^7\right) = \infty$$

$$\lim_{x\to\infty} f(x) = -\infty$$
$$\lim_{x\to-\infty} f(x) = \infty$$

Sketching Polynomials

To sketch a polynomial, you must determine at least two things for the polynomial:

- Zeros with multiplicity of the polynomial (by factoring, which may involve long division of polynomials)
  - Use multiplicity of each zero to determine if polynomial crosses x-axis at the zero (i.e., changes sign).
- End behaviour (by examining leading term)

You may also choose to label points of interest, such as y-intercepts. Finding the exact location of extrema for higher degree polynomials will require calculus.

Example Sketch the graph of the polynomial $g(x) = x(9 - 6x + x^2)(9x^2 - 24x + 16)$ by hand.

This is a fifth degree polynomial, so it will have at most 5 real valued roots.
To determine all the roots, we need to see if we can factor the two quadratics any further.

$$\begin{aligned} 9 - 6x + x^2 &= x^2 - 6x + 9 = (x-3)(x-3) = (x-3)^2 && \text{factored by inspection} \\ 9x^2 - 24x + 16 &= (3x-4)(3x-4) = (3x-4)^2 && \text{factored by inspection} \end{aligned}$$

$$g(x) = x(x-3)^2(3x-4)^2$$

The polynomial will have three zeros, at $x = 0, 4/3, 3$. Multiplicities:

- zero at $x = 0$ has multiplicity 1 (odd) so g changes sign,
- zero at $x = 4/3$ has multiplicity 2 (even) so g does not change sign,
- zero at $x = 3$ has multiplicity 2 (even) so g does not changes sign.

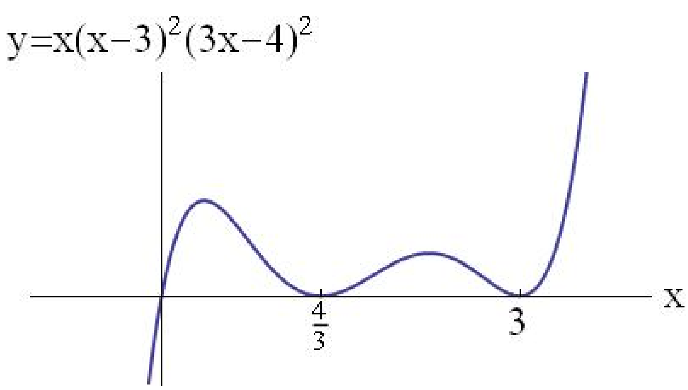
The end behaviour of the polynomial is found by determining the leading term, which is

$$x(x-3)^2(3x-4)^2 \sim x x^2(3x)^2 = 9x^5 \text{ for large } |x| \ .$$

Note "for large $|x|$" here is shorthand for saying "for x large positive or x large negative".
The end behaviour of the monomial $9x^5$ is

$$\lim_{x\to-\infty} 9x^5 = -\infty \qquad\qquad \lim_{x\to\infty} 9x^5 = \infty$$

From all this, we can sketch the graph of $g(x)$ without turning on a machine.



Example Sketch the graph of the polynomial $g(x) = x(x+3)^3(x-3)^2(2-x)$ by hand.

This is a seventh degree polynomial, so it will have at most 7 real valued roots.
The polynomial is already factored, which saves us a lot of work!

The polynomial will have four zeros, at $x = -3, 0, 2, 3$. Multiplicities:

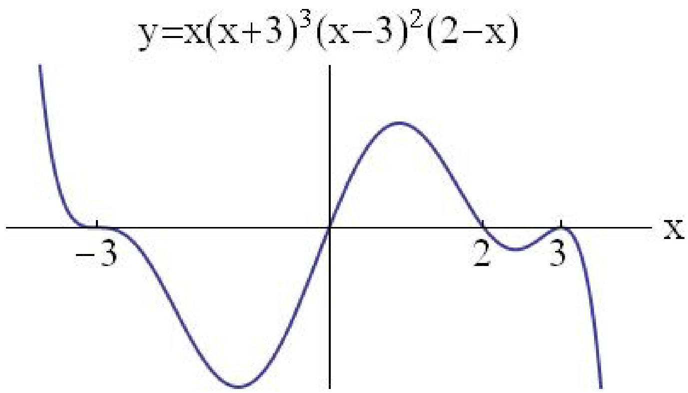
- zero at $x = -3$ has multiplicity 3 (odd) so g changes sign (since multiplicity is greater than 2, g will be horizontal at $x = -3$),
- zero at $x = 0$ has multiplicity 1 (odd) so g changes sign,
- zero at $x = 2$ has multiplicity 1 (odd) so g changes sign,
- zero at $x = 3$ has multiplicity 2 (even) so g does not changes sign.

The end behaviour of the polynomial is found by determining the leading term, which is

$$x(x+3)^3(x-3)^2(2-x) \sim x(x)^3(x)^2(-x) = -x^7 \text{ for large } |x|.$$

The end behaviour of the monomial $-x^7$ is

$$\lim_{x\to-\infty} -x^7 = \infty \qquad \lim_{x\to\infty} -x^7 = -\infty$$



Example Sketch by hand the function $f(x) = 2x^4 - 11x^3 + 22x^2 - 19x + 6$, given $x = 1$ is a root of multiplicity 2 of f.
Since $x = 1$ is a root, we can factor it out using long division.

$$
\begin{array}{r}
2x^3 - 9x^2 + 13x - 6 \\
x-1 \,\big)\, \overline{2x^4 - 11x^3 + 22x^2 - 19x + 6} \\
\underline{2x^4 - 2x^3} \\
-9x^3 + 22x^2 - 19x + 6 \\
\underline{-9x^3 + 9x^2} \\
13x^2 - 19x + 6 \\
\underline{13x^2 - 13x} \\
-6x + 6 \\
\underline{-6x + 6} \\
0
\end{array}
\qquad
\begin{array}{r}
2x^2 - 7x + 6 \\
x-1 \,\big)\, \overline{2x^3 - 9x^2 + 13x - 6} \\
\underline{2x^3 - 2x^2} \\
-7x^2 + 13x - 6 \\
\underline{-7x^2 + 7x} \\
6x - 6 \\
\underline{6x - 6} \\
0
\end{array}
$$

$$f(x) = 2x^4 - 11x^3 + 22x^2 - 19x + 6 = (x-1)(2x^3 - 9x^2 + 13x - 6) = (x-1)g(x)$$

Since $x = 1$ is a root of multiplicity 2, it can be factored out of $g(x)$! Let's factor it out using long division (details above.

$$g(x) = 2x^3 - 9x^2 + 13x - 6 = (x-1)(2x^2 - 7x + 6)$$

$$f(x) = (x-1)g(x) = (x-1)^2(2x^2 - 7x + 6)$$

We can do the factoring of the quadratic by inspection, $f(x) = (x-1)^2(2x-3)(x-2)$. Now we can sketch, since the factoring is done.

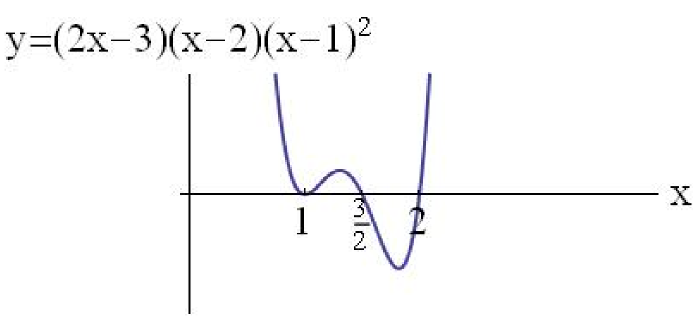
The polynomial will have three zeros, at $x = 1, 3/2, 2$. Multiplicities:

- zero at $x = 1$ has multiplicity 2 (even) so f does not change sign,
- zero at $x = 3/2$ has multiplicity 1 (odd) so f changes sign,
- zero at $x = 2$ has multiplicity 1 (odd) so f changes sign.

The end behaviour of the polynomial is found by determining the leading term, which is

$$(x-1)^2(2x-3)(x-2) \sim (x)^2(2x)(x) = 2x^4 \text{ for large } |x|.$$

The end behaviour of the monomial $2x^4$ is $\lim_{x\to-\infty} 2x^4 = \infty \qquad \lim_{x\to\infty} 2x^4 = \infty$.



Solving Polynomial Inequalities using Sign Charts

A sign chart simply lists the sign of the function you are interested in along the x-axis. It does not provide as much information as a sketch, but it provides enough to solve inequalities.

Constructing a sign chart for a polynomial relies on knowing the *end behaviour* of the polynomial (which we get from the leading coefficient), and the x-intercepts. We will see more complicated sign charts for rational functions later.

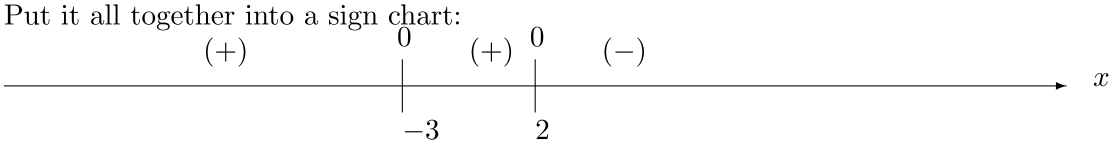
Note: The text describes sign charts slightly differently, in that they use a test-point method to determine the sign of the quadratic in each interval once the x-intercepts are known. Therefore, they do not need to know the end behaviour when they construct their sign chart. The technique I have outlined below and the one the text uses are both common techniques to generate sign charts and solve inequalities.

Example Solve the inequality $f(x) = -2(x-2)^3(x+3)^2 < 0$ by constructing a sign chart by considering x-intercepts, multiplicity, and end behaviour.

The x-intercepts are at $x = 2$ and $x = -3$. The $x = 2$ intercept has multiplicity 3, which is odd, so the function f will cross the x-axis here (change sign). The $x = -3$ intercept has multiplicity 2, which is even, so the function f will not cross the x-axis here (will not change sign).

End behaviour: For $|x|$ large, $f(x) = -2(x-2)^3(x+3)^2 \sim -2(x)^3(x^2) = -2x^5$. This monomial will approach $-\infty$ if x is large, due to the minus sign of the coefficient. As x approaches $-\infty$, the monomial will approach ∞.

Put it all together into a sign chart:



From the sign diagram, we see that $f(x) < 0$ if $x \in (2, \infty)$.