> **Concepts:** Horizontal Asymptotes, Vertical Asymptotes, Slant (Oblique) Asymptotes, Transforming Reciprocal Function, Sketching Rational Functions, Solving Inequalities using Sign Charts.

Rational Function

A *rational function* is a function which is a ratio of two polynomials g and h: $f(x) = \dfrac{g(x)}{h(x)}$, where $h(x) \neq 0$.

Since polynomials are continuous functions, the domain of a rational function is all $x \in \mathbb{R}$ except possibly at values of x for which the denominator is zero.

Vertical Asymptotes

Where the denominator goes to zero it is possible to have a vertical asymptote of the rational function. The values of x that make the denominator zero are not in the domain of the rational function.

The function f has a *vertical asymptote* at $x = a$ if:

$$\lim_{x \to a^-} f(x) = \pm\infty, \qquad \text{or} \qquad \lim_{x \to a^+} f(x) = \pm\infty.$$

New notation: Limits from the left and right:
$\lim\limits_{x \to a^-} f(x)$ means the limit as x approaches a from the left (negative side of the x axis from a),
$\lim\limits_{x \to a^+} f(x)$ means the limit as x approaches a from the right (positive side of the x axis from a).

Horizontal and Slant Asymptotes

Horizontal and slant (or oblique) asymptotes are straight lines the function approaches as $|x|$ gets very, very large.

The function f has a *horizontal asymptote* $y = L$ if:

$$\lim_{x \to \infty} f(x) = L, \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L.$$

The function f has a *slant asymptote* $y = mx + b$ if:

$$\lim_{x \to \infty} f(x) = mx + b, \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = mx + b.$$

To get horizontal and slant asymptotes <u>algebraically</u> we need to know about end behaviour for rational functions.

- If the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator, f has a horizontal asymptote $y = 0$.

- If the degree of the polynomial in numerator is equal to the degree of the polynomial in the denominator, f has a horizontal asymptote $y = L$.

Technique: To determine the horizontal asymptote of a rational function, divide the numerator and denominator by the highest power of x in the expression.

- If the degree of the polynomial in numerator is one more than the degree of the polynomial in the denominator, f has a slant asymptote $y = mx + b$.

Technique: To determine the slant asymptote of a rational function, use long division of polynomials to simplify the rational function.

Example Determine the end behaviour of the function $f(x) = \dfrac{3x^3 + 2x + 1}{5x^3 + 3x^2 + 1}$.

The highest power of x is x^3, so divide everything by x^3:

$$f(x) = \frac{3x^3 + 2x + 1}{5x^3 + 3x^2 + 1} = \frac{\frac{3x^3}{x^3} + \frac{2x}{x^3} + \frac{1}{x^3}}{\frac{5x^3}{x^3} + \frac{3x^2}{x^3} + \frac{1}{x^3}} = \frac{3 + \frac{2}{x^2} + \frac{1}{x^3}}{5 + \frac{3}{x} + \frac{1}{x^3}}$$

Now, as $|x|$ gets very large, the terms that are divided by x get very small (negligible). From this, we can see that

$$\lim_{x \to \pm\infty} f(x) = \frac{3}{5}$$

so we see that f has a horizontal asymptote of $y = 3/5$.

Example Determine the end behaviour of the function $f(x) = \dfrac{x^2 + 1}{x - 9}$.

$$
\begin{array}{r}
x + 9 \\
x - 9 \overline{\smash{\big)}\, x^2 + 0x + 1} \\
\underline{x^2 - 9x} \quad \text{(subtract)} \\
9x + 1 \\
\underline{9x - 81} \quad \text{(subtract)} \\
82
\end{array}
$$

$\Rightarrow \quad \dfrac{x^2 + 1}{x - 9} = x + 9 + \dfrac{82}{x - 9}$

for $|x|$ very large, $\dfrac{82}{8 - 9}$ is negligible.

Since $f(x) = \frac{x^2 + 1}{x - 9} = x + 9 + \frac{82}{x - 9}$, we see that as $|x|$ gets very large the $\frac{82}{x - 9}$ goes to zero, so there is a slant asymptote $y = x + 9$.

Example Consider the sketch of the function $g(x) = \dfrac{3x^2}{4 - x^2}$. Identify the horizontal and vertical asymptotes from the sketch.
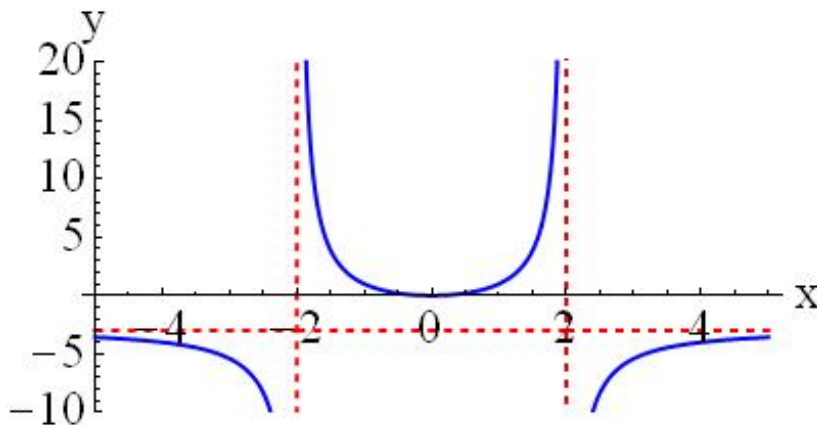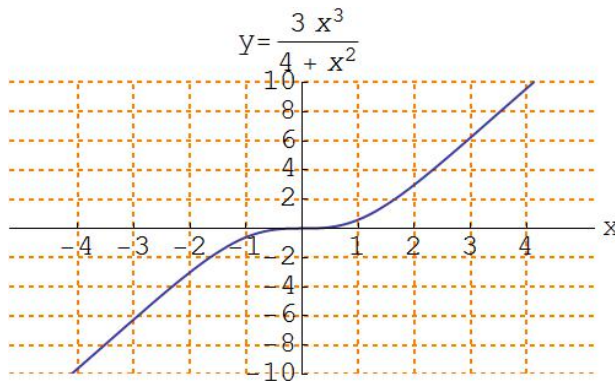

Vertical Asymptotes: $x = -2$ and $x = 2$. Horizontal Asymptotes: $y = -3$.

Example What does the function $g(x) = \dfrac{3x^3}{4 + x^2}$ look like if x is very large?

Graphical Solution

Let's get a sketch using a computer or calculator, and see what we can learn.

From the sketch, the graph appears to be linear for large x. We new this would be the case since the degree of the polynomial in the numerator is one more than the degree of the polynomial in the denominator.

Long division of polynomials will show you that $y = 3x - \frac{12x}{4+x^2}$, so the slant asymptote is $y = 3x$. In this case, the slant asymptote passes through the origin (this is not always the case).

Here's a sketch, with the slant asymptote shown as the dashed line.
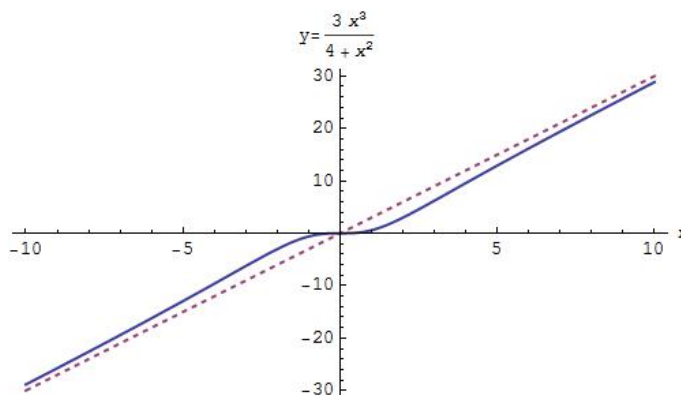
Sketching a Ratio of Linear Functions by Transforming the Reciprocal Function

If the rational function f is a ratio of <u>linear functions</u>, $f(x)$ can be sketched by finding how f is transformed from the reciprocal function.

Although this is an interesting use of the ideas of graphical transformation, it is usually easier to just use the techniques for sketching a rational function in general. I include this example solely as a example of graphical transformations in action.

Example Sketch the graph of $f(x) = \dfrac{2x - 5}{6x - 12}$ by transforming the reciprocal function.

First, we must use long division to write $f(x)$ with a degree 1 polynomial in the denominator and a constant in the numerator

$$
\begin{array}{r}
\frac{1}{3} \\
6x-12 \overline{)\,2x-5} \\
\underline{2x-4} \\
-1
\end{array}
$$

So $f(x) = \dfrac{2x - 5}{6x - 12} = \dfrac{1}{3} - \dfrac{1}{6x - 12}$. Now, we need to find the transformations of $g(x) = \dfrac{1}{x}$ that lead to $f(x)$:

Basic function: $y = g(x) = \dfrac{1}{x}$.

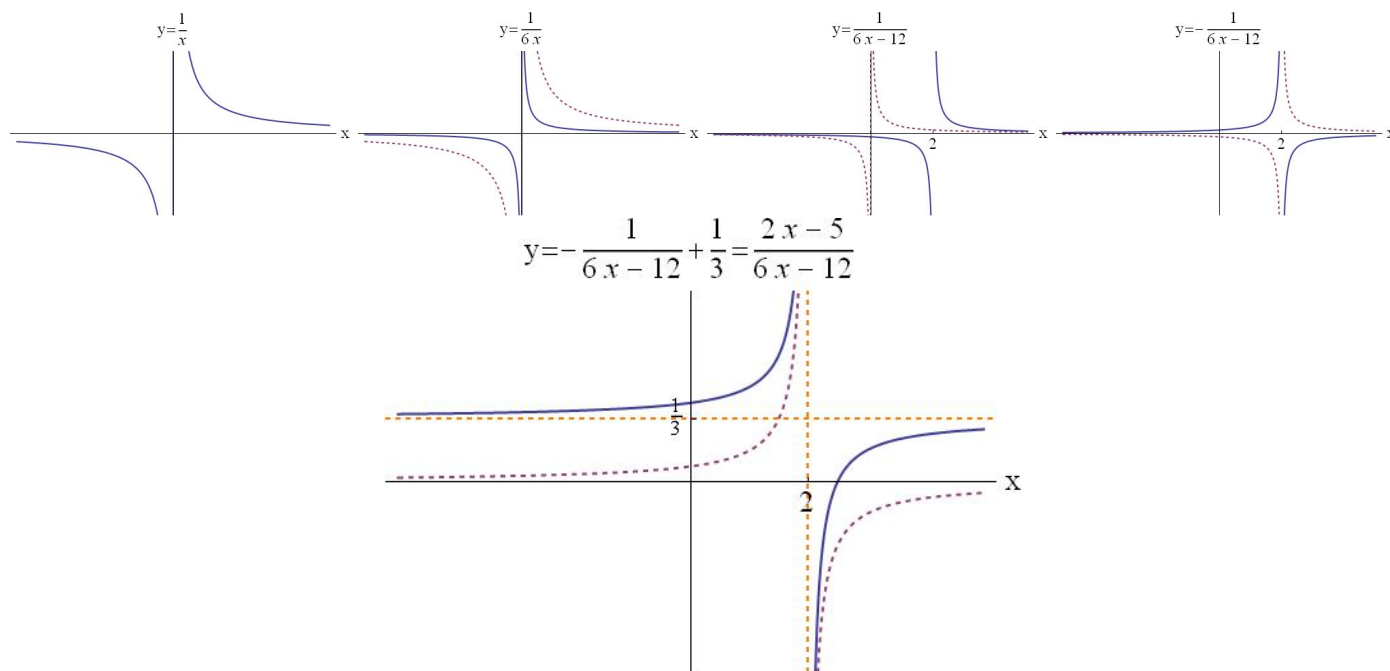Compress horizontally by 6 units: $y = g(6x) = \dfrac{1}{6x}$.

Shift right by 2 units: $y = g(6(x - 2)) = \dfrac{1}{6x - 12}$.

Rotate about the x-axis: $y = -g(6x - 12) = -\dfrac{1}{6x - 12}$.

Shift up 1/3 units: $y = -g(6x - 12) + \dfrac{1}{3} = -\dfrac{1}{6x - 12} + \dfrac{1}{3} = f(x)$.

and we have found the transformations of $y = 1/x$ which lead to $f(x)$.

Sketches (I have included the previous sketch (dashed line) in the set of transformations for reference purposes only):

Recall: Graphing a Polynomial $f(x)$ by Hand

Polynomial functions can be sketched by hand if you do the following:

1) Examine end behaviour,
2) Find any zeros (x-intercepts) (factor the polynomial if possible, get multiplicities),
3) Find the y-intercept, which is $f(0)$ (it might be a point of interest).

Graphing a Rational Function $f(x)$ by Hand

Rational functions can be sketched by hand if you do the following:

1) Examine end behaviour (look for horizontal asymptotes, slant asymptotes),
2) Look for vertical asymptotes (factor the denominator if possible, get multiplicities),
3) Find any zeros (x-intercepts) (factor the numerator if possible, get multiplicities),
4) Find the y-intercept, which is $f(0)$ (it might be a point of interest).

Example Sketch the graph of $f(x) = \dfrac{2x - 5}{6x - 12}$.

The function will have one zero, at $x = 5/2$. Multiplicities:

- zero at $x = 5/2$ has multiplicity 1 (odd) so f changes sign,

The end behaviour is found by determining the leading term in the numerator and denominator, which is

$$\frac{2x - 5}{6x - 12} \sim \frac{2x}{6x} = \frac{1}{3} \text{ for large } |x|.$$

The function $f(x)$ has a horizontal asymptote $y = 1/3$.

The function will have one vertical asymptote, where $6x - 12 = 0$, at $x = 2$. Multiplicities:

- vertical asymptote at $x = 2$ has multiplicity 1 (odd) so f changes sign.

We now know all the interesting features of the graph!

Sketch by starting on the far right, where we know there is a horizontal asymptote $y = 1/3$.

As you move to the left, the function must cross the x-axis at the zero, $x = 5/2$. It then approaches minus infinity at the vertical asymptote $x = 2$.

The function changes sign at the vertical asymptote, so on the other side it begins up at infinity and drops down to approach the horizontal asymptote $y = 1/3$ again.
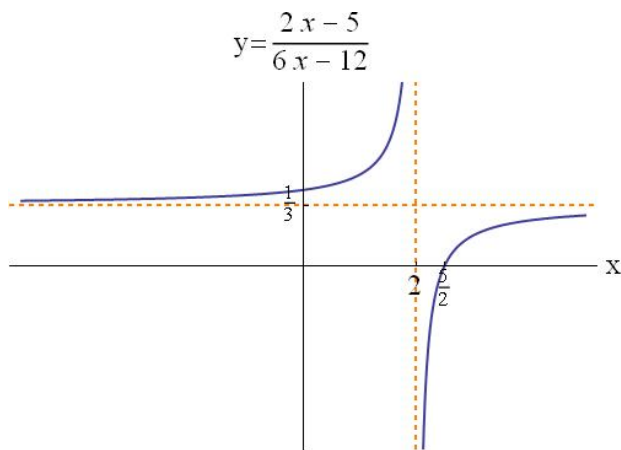
I think this was easier than transforming the reciprocal function.

From the sketch, we can, if we wanted to, write things like the following:

$$\text{Vertical Asymptote:} \quad \lim_{x \to 2^+} f(x) = -\infty$$
$$\lim_{x \to 2^-} f(x) = +\infty$$
$$\text{Horizontal Asymptote:} \quad \lim_{x \to \infty} f(x) = 1/3$$
$$\lim_{x \to -\infty} f(x) = 1/3$$

Example Sketch by hand the graph of $f(x) = \dfrac{h(x)}{g(x)} = \dfrac{3x^3 + x - 4}{x^3 + 1}$.

Examine the end behaviour (the <u>leading term</u> is dominant in the numerator and denominator):

$$f(x) = \frac{3x^3 + x - 4}{x^3 + 1} \sim \frac{3x^3}{x^3} = 3 \text{ if } |x| \text{ is large.}$$

The function $f(x)$ has a horizontal asymptote $y = 3$, since $\lim\limits_{x \to \infty} f(x) = 3$ and $\lim\limits_{x \to -\infty} f(x) = 3$.

Factor the denominator:

$$g(x) = x^3 + 1$$

This is a sum of cubes, so use the formula for sum of cubes to get $g(x) = x^3 + 1^3 = (x+1)(x^2 - 1 \cdot x + 1^2) = (x+1)(x^2 - x + 1)$. If you don't notice this is a sum of cubes (you should), you can always use the ideas from section on Real Zeros of Polynomials to factor.

We know the quadratic $x^2 - x + 1$ will not factor, since we know $g(x) = x^3 + 1$ is simply the cubing function shifted up 1 unit, and so it will only have one x-intercept. Alternately, we could check that $b^2 - 4ac = (-1)^2 - 4(1)(1) = -3 < 0$ so the quadratic has no real roots.
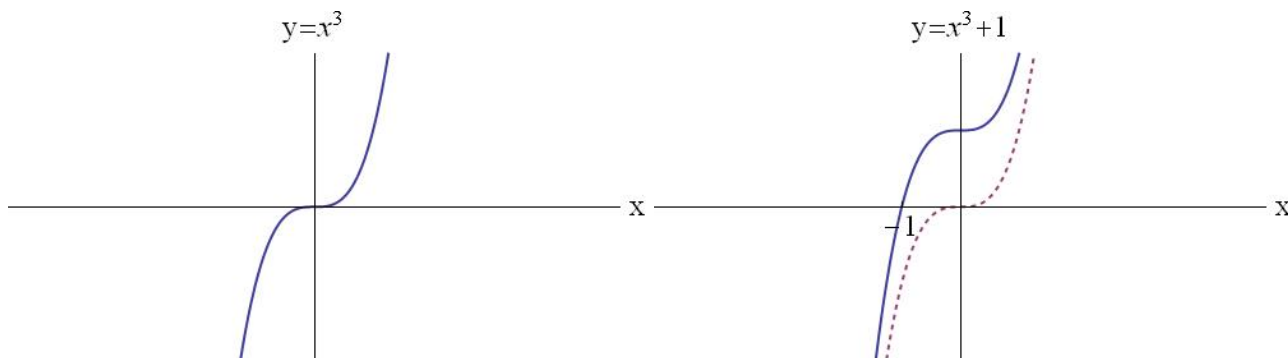
We now need to factor the numerator:

$$h(x) = 3x^3 + x - 4$$

We see by inspection that $h(1) = 0$, so $h(x)$ has a root $x = 1$. Let's factor $(x - 1)$ out:

$$h(x) = 3x^3 + x - 4 = (x - 1)(3x^2 + 3x + 4)$$

The quadratic in this case has $b^2 - 4ac = (3)^2 - 4(3)(4) = -39 < 0$ so the quadratic has no real roots.

What we've learned from the factoring is that we can write

$$f(x) = \frac{h(x)}{g(x)} = \frac{3x^3 + x - 4}{x^3 + 1} = \frac{(x - 1)(3x^2 + 3x + 4)}{(x + 1)(x^2 - x + 1)}$$

The function $f(x)$ has an x-intercept at $x = 1$, since $f(1) = 0$. Since this root is odd multiplicity (1), f will change sign at $x = 1$.

There is a vertical asymptote at $x = -1$. Since this vertical asymptote has odd multiplicity (1), f will f will change sign at $x = -1$.

The y-intercept is $f(0) = \dfrac{3(0)^3 + (0) - 4}{(0)^3 + 1} = -4$.

We now know all the interesting features of the graph!

Sketch by starting on the far right, where we know there is a horizontal asymptote $y = 3$.

As you move to the left, the function must cross the x-axis at the zero, $x = 1$. It then passes through the y-intercept and approaches minus infinity at the vertical asymptote $x = -1$.

The function changes sign at the vertical asymptote, so on the other side it begins up at infinity and drops down to approach the horizontal asymptote $y = 3$ again.
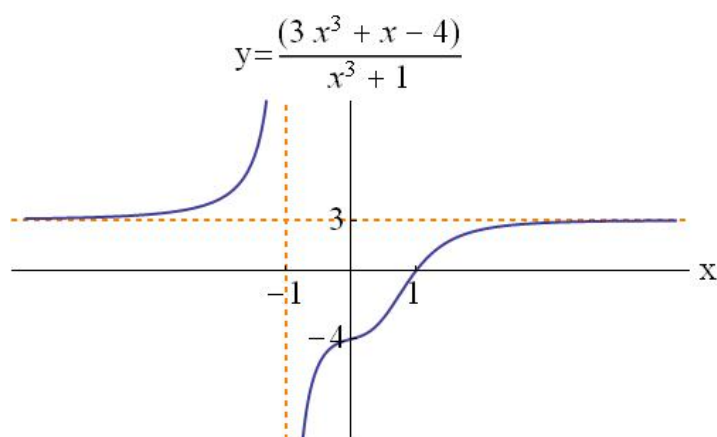
From the sketch, we can write things like the following:

$$\text{Vertical Asymptote:} \quad \lim_{x \to -1^+} f(x) = -\infty$$
$$\lim_{x \to -1^-} f(x) = +\infty$$
$$\text{Horizontal Asymptote:} \quad \lim_{x \to \infty} f(x) = 3$$
$$\lim_{x \to -\infty} f(x) = 3$$

Solving Rational Inequalities using Sign Charts

A sign chart simply lists the sign of the function you are interested in along the x-axis. It does not provide as much information as a sketch, but it provides enough to solve inequalities.

Constructing a sign chart for a rational inequalities relies on knowing the end behaviour, the vertical asymptotes (with multiplicity), and the x-intercepts (with multiplicity).

We only need these things since the only place the rational function can change sign is at vertical asymptotes and x-intercepts.

Note: The text describes sign charts slightly differently, in that they use a test-point method to determine the sign of the quadratic in each interval once the x-intercepts are known. Therefore, they do not need to know the end behaviour when they construct their sign chart. The technique I have outlined below and the one the text uses are both common techniques to generate sign charts and solve inequalities.

Example Solve the inequality $\dfrac{x^2 + 2x + 1}{x^2 - 2x - 15} \geq 0$.

$f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 2x - 15} = \dfrac{(x+1)^2}{(x-5)(x+3)}$ Factor numerator and denominator.

End Behavior

$f(x) = \dfrac{1 + \frac{2}{x} + \frac{1}{x^2}}{1 - \frac{2}{x} - \frac{15}{x^2}}$ So $\lim\limits_{x \to \infty} f(x) = \dfrac{1}{1} = 1 > 0$ ← All we need for sign chart is if this is positive or negative.
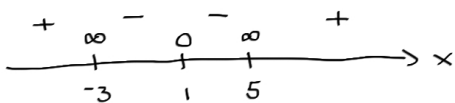
Zeros

$x = -1$, multiplicity 2 $\rightarrow$ f does not change sign at $x = -1$

Vertical Asymptotes

$x = 5, -3$, multiplicity 1 $\rightarrow$ f changes sign at $x = 5, -3$.

Sign chart

$\begin{array}{ccccccc} + & & - & & - & & + \\ & \infty & & 0 & & \infty & \\ \hline & | & & | & & | & \to x \\ & -3 & & 1 & & 5 & \end{array}$

$f(x) \geq 0$ for

$x \in (-\infty, -3) \cup [1,1] \cup (5, \infty)$.

note $[1,1]$ could be written $\{1\}$.