Concepts: Exponential Functions, base b, the base e, properties of exponential functions (domain, range, increasing, decreasing), graphical transformations, solving simple exponential equations, continuous compound interest, radioactive decay.

Laws of Exponents To work algebraically with exponential functions, we need to use the laws of exponents. You should memorize these laws.

If x and y are real numbers, and $b > 0$ is real, then

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$

Exponential Functions

A function of the form $f(x) = a \cdot b^x$ is an *exponential function*, where $a \neq 0$ and $b > 0, b \neq 1$ are real numbers. The number b is called the base of the exponential function.

Note: The difference between a monomial and an exponential is where the variable is.
monomial: $f(x) = x^3$ has variable x in base.
exponential: $f(x) = 3^x$ has variable x in the exponent.
constant: $f(x) = 3^\pi$ is a constant function (no x).
more complicated function: $f(x) = x^x$ has an x in the base and in the exponent (you will see these in calculus).

Exponential functions are defined for all real numbers. For some real numbers, it is easy to figure out what the exponential function is. Consider $f(x) = 3^x$, which is an exponential function.

Evaluate at an integer: $f(4) = 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$.

Evaluate at zero: $f(0) = 3^0 = 1$.

Evaluate at negative integer: $f(-4) = 3^{-4} = \frac{1}{3^4} = \frac{1}{81} \sim 0.012345$.

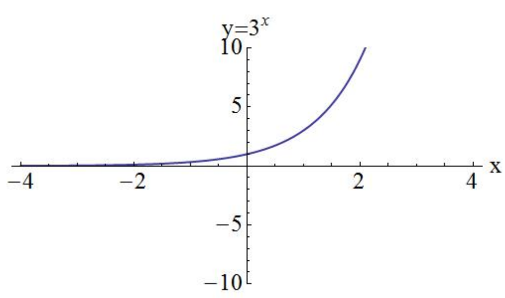
Evaluate at a rational number: $f(-3/2) = 3^{-3/2} = \frac{1}{3^{3/2}} = \frac{1}{\sqrt{27}} \sim 0.19245$.

However, we cannot so easily figure out what an exponential function is when we evaluate it at a irrational number, for example what is $f(\pi) = 3^\pi$.

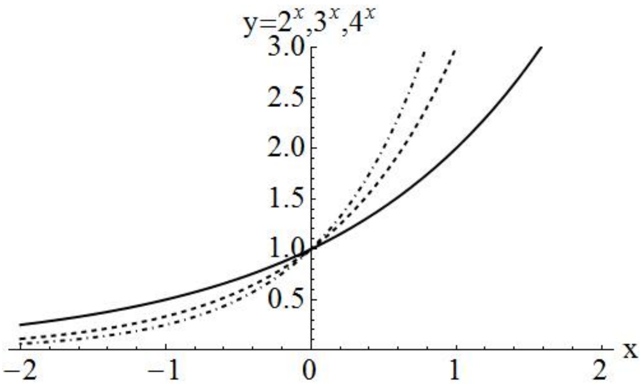
We can determine the value of the number 3^π by a procedure which will get us as close as we want to the number:

$$
\begin{aligned}
3^\pi &\sim 3^3 = 27 \\
3^\pi &\sim 3^{31/10} = 30.1353 \\
3^\pi &\sim 3^{314/100} = 31.4891 \\
3^\pi &\sim 3^{314159/100000} = 31.5442 \\
3^\pi &\sim 31.5443
\end{aligned}
$$

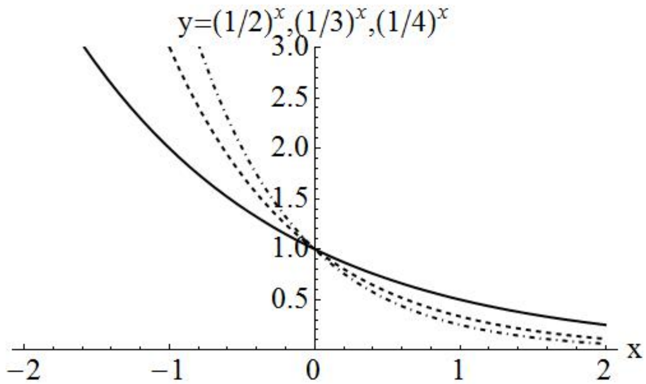
In this way, we can evaluate an exponential at any real value for x. The sketch of the exponential function looks like:



Sketches with different bases:



Solid: $y = 2^x$, Dashed: $y = 3^x$, DotDashed: $y = 4^x$



Solid: $y = (1/2)^x$, Dashed: $y = (1/3)^x$, DotDashed: $y = (1/4)^x$

Growth or Decay (Increasing or Decreasing)

If $a > 0$ and $b > 1$, then $f(x) = a \cdot b^x$ is increasing and called an *exponential growth function* (b is called the *growth factor*).

If $a > 0$ and $0 < b < 1$, then $f(x) = a \cdot b^x$ is decreasing and called an *exponential decay function* (b is called the *decay factor*).

This is also sometimes stated the following way, since for $b > 1$ the quantity $1/b$ will be less than one, so we have exponential decay with $\left(\frac{1}{b}\right)^x = (b^{-1})^x = b^{-x}$, and we can say:

If $a > 0$ and $b > 1$, then $a \cdot b^x$ is increasing, and $a \cdot b^{-x}$ is decreasing.

Example Determine the exponential function that passes through the points $(0, 5)$ and $(2, 16)$.

The general exponential function is $f(x) = a \cdot b^x$. We can use the two points we are given to determine the two constants a and b.

$$f(0) = 5 = a \cdot b^0 = a$$

so $a = 5$.

$$f(2) = 16 = 5 \cdot b^2$$

Solve this equation.

$$\begin{aligned} 16 &= 5 \cdot b^2 \\ \frac{16}{5} &= b^2 \\ \left(\frac{16}{5}\right)^{1/2} &= (b^2)^{1/2} \\ \sqrt{\frac{16}{5}} &= b \end{aligned}$$

The exponential function that passes through the two points is $y = f(x) = 5(\sqrt{16/5})^x$. We do not use the negative root for b since for exponential functions the base must be greater than zero.

Compound Interest Formula

Compound interest pays interest on the principal P (the initial amount deposited) and the accumulated interest, not just the principal. I want to take the time to motivate where the compound interest formula comes from.

Here is the basic formula (in English) for compound interest:

accumulated amount = previous accumulated amount+previous accumulated amount·interest rate per compounding period

For a nominal annual interest rate r, compounded n times per year, we have $i = r/n$ as the *interest rate per compounding period*. Now let's try to derive a formula for compound interest.

Compounding Period	Accumulated Amount
0	P
1	$P + Pi = P(1+i)$
2	$P(1+i) + P(1+i)i = P(1+i)^2$
3	$P(1+i)^2 + P(1+i)^2 i = P(1+i)^3$
4	$P(1+i)^3 + P(1+i)^3 i = P(1+i)^4$
	$\vdots$
m	$P(1+i)^m$

ie., for a principal of P with compound interest of $i = r/n$ paid every compounding period, we get an accumulated amount after $m = nt$ compounding periods (t is number of years, n is number of compounding periods per year) of

$$A = P(1+i)^m = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Example \$1000 is deposited at 7.5% per year. Find the balance at the end of one year, and two years, if the interest paid is compounded daily.
Solution
The nominal annual rate is r =7.5% = 0.075, when compounded daily, means we have $n = 365$, so $i = r/n = 0.075/365 = 0.000205479$.
One year corresponds to $m = nt = 365 \times 1 = 365$, so after one year we have

$$A = P(1+i)^m = \$1000.00(1 + 0.000205479)^{365} = \$1077.88.$$

Two years corresponds to $m = nt = 365 \times 2 = 730$, so after two years we have

$$A = P(1+i)^m = \$1000.00(1 + 0.000205479)^{730} = \$1161.82.$$

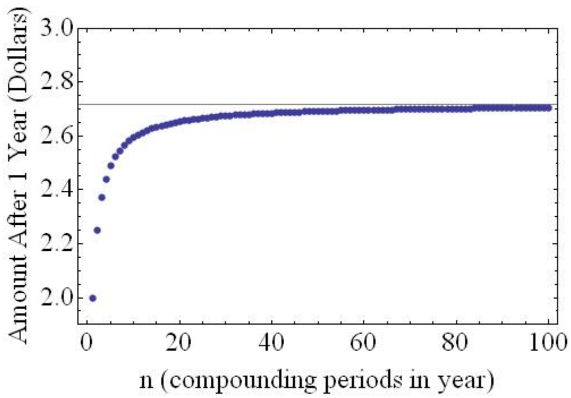
Compounding Continuously

Consider a principal $P = \$1$ and a rate of r=100% which is compounded over shorter and shorter time periods. We are interested in how much the accumulated amount will be after one year.

Compound interest compounded n times a year ($i = 1/n$, and $m = nt = n$ (to get one year, $t = 1$)):

$$A = P(1+i)^m = \left(1 + \frac{1}{n}\right)^n \text{ after 1 year.}$$

Here is a sketch



We see that the accumulated amount is approaching a number:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \sim 2.71828\ldots$$

This number is similar to $\pi = 3.14...$ in that it is mathematically significant, appears in many situations, and is a nonrepeating nonterminating decimal and so we give it a special designation

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \sim 2.71828\ldots$$

This leads the the continuous interest formula, which is

$A = Pe^{rt}$ after t years if interest is compounded continuously at annual rate r.

The continuous interest formula is the upper limit on the accumulated amount that can accrue due to compounding interest.

The base e

This base e is a preferred base in calculus (you will see why in calculus). It is an irrational number $e = 2.718281828\ldots$. Our analysis above for continuously compounding interest can be used to define

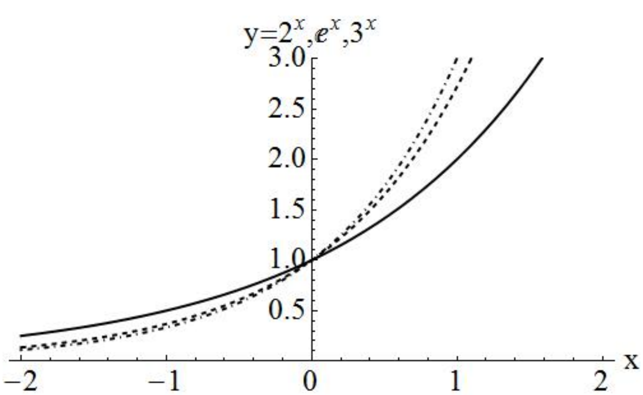
$$e^{rt} = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$$

For now, remember that e is simply an irrational number, and it is the preferred base for the exponential function.

The Natural Exponential Function

$$f(x) = a \cdot e^{kx}$$

If $a > 0$ and $k > 0$, this is an exponential growth function.
If $a > 0$ and $k < 0$, this is an exponential decay function.



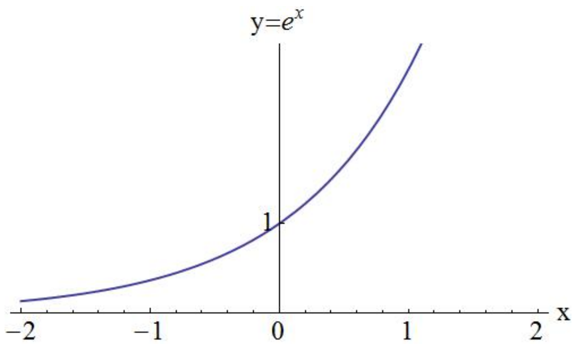
Once we learn about logarithms, we shall see that we can always convert from a general base b to base e using $b^x = e^{kx}$. Logarithms will allow to determine the value of k for a given b, but even without logarithms we can see this is true using the rules of exponents:

$$\begin{aligned} f(x) &= b^x \text{ (rewrite } b = e^k \text{ for some value of } k) \\ &= (e^k)^x \\ &= e^{kx} \end{aligned}$$

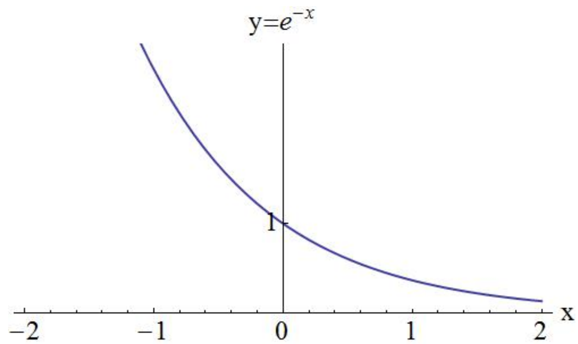
For this reason, I tend to focus on $y = e^{kx}$ for exponential growth and $y = e^{-kx}$ for exponential decay, although we shall see that when modeling with exponentials it is often more beneficial to use a different base.

The exponential function can be transformed using our typical transformation techniques (for example, $y = 2^{x+3}$ is $y = 2^x$ shifted to the left by three units).

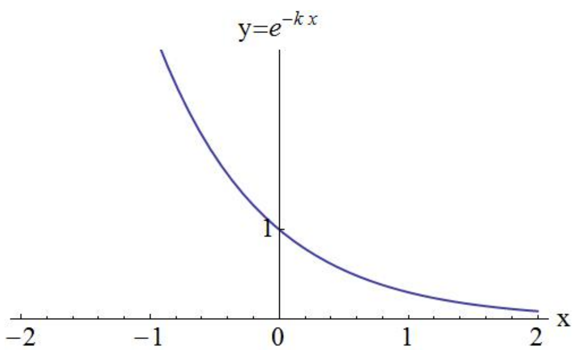
Example Sketch $y = ae^{-kx}$ where $a > 0$ and $k > 0$ are real numbers using graphical transformations of $y = e^x$.



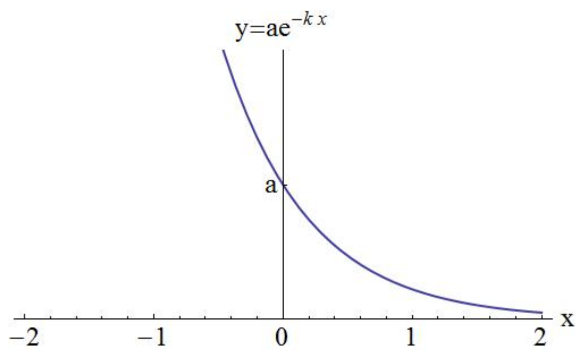
Basic Function $y = f(x) = e^x$



$y = f(-x) = e^{-x}$ reflect about y-axis



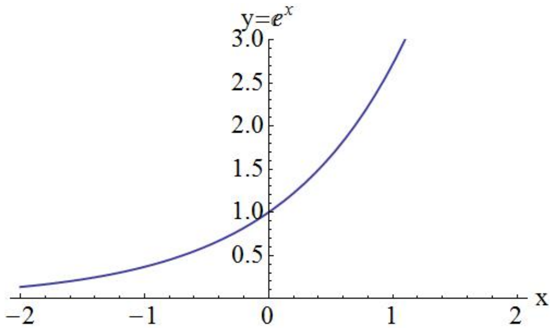
$y = f(-kx) = e^{-kx}$ horizontal compression by factor of k



$y = af(-kx) = ae^{-kx}$ vertical stretch by a factor of a

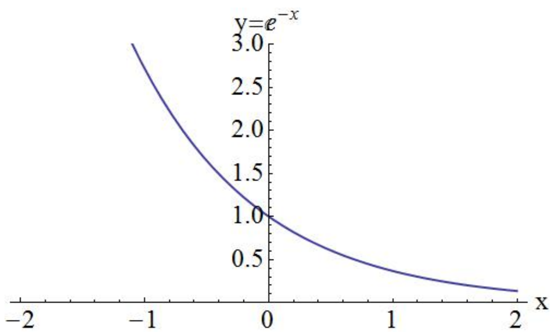
From the sketch we can analyze the behaviour of the function $f(x) = ae^{-kx}$ where $a > 0$ and $k > 0$:
Domain: $x \in \mathbb{R}$
Range: $y \in (0, \infty)$
Continuous
Neither odd nor even
Bounded below, but not above
No local extrema
Horizontal Asymptote: $y = 0$
No Vertical Asymptotes
Decreasing
End Behaviour: $\lim\limits_{x\to-\infty} ae^{-kx} = \infty$ and $\lim\limits_{x\to\infty} ae^{-kx} = 0$

Properties of $y = e^x$



Domain: $x \in \mathbb{R}$
Range: $y \in (0, \infty)$
Continuous
Neither odd nor even
Bounded below, but not above
No local extrema
Horizontal Asymptote: $y = 0$
No Vertical Asymptotes
Increasing
End Behaviour: $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$

Properties of $y = e^{-x}$



Domain: $x \in \mathbb{R}$
Range: $y \in (0, \infty)$
Continuous
Neither odd nor even
Bounded below, but not above
No local extrema
Horizontal Asymptote: $y = 0$
No Vertical Asymptotes
Decreasing
End Behaviour: $\lim_{x \to -\infty} e^{-x} = \infty$ and $\lim_{x \to \infty} e^{-x} = 0$

Radioactive Decay

Radioactive decay is modeled by the function $A(t) = A_0 e^{rt}$ which gives the amount of radioactive material $A(t)$ remaining after t years where A_0 is the initial amount and r is the annual rate of decay rate (so $r < 0$).

Radioactive decay problems will often give the *half-life* (the time it takes for a given amount to decay to half that amount) for the material rather than the annual decay rate r.

To work effectively with radioactive decay models, we need logarithms, so we will postpone looking at these types of models for the moment.