Concepts: rules of logarithms, change of base, solving equations.

When working with polynomial, rational, and radical functions, the algebraic techniques we needed to be proficient with to perform manipulations on the functions were

- finding common denominator
- factoring
- long division of polynomials
- completing the square
- rationalizing numerator or denominator

among others.

To perform manipulations on trigonometric functions, we need to be proficient with trigonometric identities. That is why trig identities are a big part of Math 1013 Precalculus II Trig.

To perform manipulations on exponential and logarithmic functions, we need to be proficient with the rules of exponents and the rules of logarithms. So these rules should be memorized, since they will form the basis of the techniques you will use when working with exponential and logarithmic functions.

The text takes the time to motivate where the rules come from.

Laws of Exponents If x and y are real numbers, and $a > 0$ is real, then

1. $a^0 = 1$
2. $a^x \cdot a^y = a^{x+y}$
3. $\dfrac{a^x}{a^y} = a^{x-y}$
4. $(a^x)^y = a^{xy}$

Laws of Logarithms If x and y are positive numbers, and $a > 0, b \neq 1$ is real, then

1. $\log_a(1) = 0$
2. $\log_a(xy) = \log_a x + \log_a y$
3. $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
4. $\log_a(x^r) = r \log_a x$ where r is any real number

Inverse Function Cancellation

1. $\log_a(a^x) = x$ for every $x \in (-\infty, \infty)$
2. $a^{\log_a(x)} = x$ for every $x \in (0, \infty)$

In calculus, you will work most frequently with the natural logarithms, so I will also give you the rules with base e^x and $\ln x$ and suggest you memorize these rules and know how to change base to base e when necessary.

Laws of Exponents If x and y are real numbers, then

1. $e^0 = 1$
2. $e^x \cdot e^y = e^{x+y}$
3. $\dfrac{e^x}{e^y} = e^{x-y}$
4. $(e^x)^y = e^{xy}$

Laws of Logarithms If x and y are positive numbers, then

1. $\ln(1) = 0$
2. $\ln(xy) = \ln x + \ln y$
3. $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$
4. $\ln(x^r) = r \ln x$ where r is any real number

Inverse Function Cancellation

1. $\ln(e^x) = x,\ \ x \in (-\infty, \infty)$
2. $e^{\ln x} = x,\ \ x > 0$

Exponential Change of Base from b to base e

You can always convert to base e, using the following application of the rules:

$$\begin{aligned} b^x &= (e^{\ln b})^x \\ &= e^{x \ln b} \end{aligned}$$

Logarithm Change from Base b to base e

This requires a bit more work, but again uses the rules:

$$\begin{aligned} y &= \log_b x \\ b^y &= b^{\log_b x} \\ b^y &= x \\ \ln(b^y) &= \ln(x) \\ y \ln(b) &= \ln(x) \\ y &= \frac{\ln(x)}{\ln(b)} \end{aligned}$$

This process can be used to change from any base to any other base.

Example Write $\log_7 x$ in terms of common and natural logarithms.

Convert to common logarithms:
Let $y = \log_7 x \longrightarrow 7^y = x$.

$$\begin{aligned} 7^y &= x \\ \log(7^y) &= \log x \\ y\log(7) &= \log x \\ y &= \frac{\log x}{\log 7} \\ \log_7 x &= \frac{\log x}{\log 7} \end{aligned}$$

Convert to natural logarithms:
Let $y = \log_7 x \longrightarrow 7^y = x$.

$$\begin{aligned} 7^y &= x \\ \ln(7^y) &= \ln x \\ y\ln(7) &= \ln x \\ y &= \frac{\ln x}{\ln 7} \\ \log_7 x &= \frac{\ln x}{\ln 7} \end{aligned}$$

Example Solve $e^{5-3x} = 10$ for x.

Take the natural logarithm of both sides:

$$\begin{aligned} \ln(e^{5-3x}) &= \ln 10 \\ 5-3x &= \ln 10 \\ -3x &= \ln 10 - 5 \\ x &= \frac{\ln 10 - 5}{-3} \\ x &= \frac{5 - \ln 10}{3} \end{aligned}$$

Example \$1000 is deposited at 7.5% per year. If the interest paid is compounded daily, how long will it take for the balance to reach \$2000?
Solution
Let t be the number of years after Jan 1.

$$\begin{aligned} P &= \$1000 \\ A &= \$2000 \\ r &= 7.5\% = 0.075 \\ n &= 365 \end{aligned}$$

The compound interest formula can be solved for t

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ A/P &= \left(1+\frac{r}{n}\right)^{nt} \\ \ln(A/P) &= \ln\left[\left(1+\frac{r}{n}\right)^{nt}\right] \\ \ln(A/P) &= nt\ln\left[1+\frac{r}{n}\right] \\ \frac{\ln A/P)}{n\ln\left[1+\frac{r}{n}\right]} &= t \\ t &= \frac{\ln(A/P)}{n\ln\left[1+\frac{r}{n}\right]} \\ &= \frac{\ln(2000/1000)}{365\ln\left[1+0.075/365\right]} = 9.24291 \end{aligned}$$

It will take 9.24291 years for the accumulated amount to reach \$2000.