Concepts: Solving Equations with Exponentials and Logarithms, exponential decay models with half-life (radioactive dating, metabolization of drugs), compound interest formula, Newton's law of cooling.

You have to be aware of extraneous solutions entering the problem when you are solving equations using exponentials and logarithms. In these cases, the extraneous solution can enter by finding a solution which is not in the domain of the original logarithmic function.

Notation: $e^x = \exp(x)$.

Technique: Try to isolate a single logarithm or exponential of x and then take a logarithm or exponential to simplify.

$$\begin{aligned}
\ln(\text{some complicated function of } x) &= \text{constant} \\
\text{some complicated function of } x &= e^{\text{constant}} \\
&\text{or} \\
e^{\text{some complicated function of } x} &= \text{constant} \\
\text{some complicated function of } x &= \ln(\text{constant})
\end{aligned}$$

In both cases, you solve for x using what we have already learned about solving equations.

Example Solve the equation $\ln(x-2) + \ln x = 2$ algebraically.

$$\begin{aligned}
\ln(x-2) + \ln x &= 2 \\
\ln((x-2)(x)) &= 2 \quad (\text{use } \ln A + \ln B = \ln(AB)\ ) \\
e^{\ln((x-2)(x))} &= e^2 \quad (\text{take exponential of both sides of equation}) \\
(x-2)(x) &= e^2 \quad (\text{simplify using } e^{\ln A} = A) \\
x^2 - 2x - e^2 &= 0 \quad (\text{quadratic in } x) \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-e^2)}}{2(1)} \\
&= \frac{2 \pm \sqrt{4 + 4e^2}}{2} \\
&= \frac{2 \pm \sqrt{4}\sqrt{1 + e^2}}{2} \\
&= \frac{2 \pm 2\sqrt{1 + e^2}}{2} \\
&= 1 \pm \sqrt{1 + e^2}
\end{aligned}$$

However, $1 - \sqrt{1+e^2} < 0$, and in the original equation we had to have $x > 2$ for the logarithms to be defined. Therefore, the only solution is $x = 1 + \sqrt{1+e^2}$.

Example Solve the equation $\ln x^2 = 2$ for x.

$$\begin{aligned}
\ln x^2 &= 2 \\
e^{\ln x^2} &= e^2 \quad (\text{take exponential of both sides}) \\
x^2 &= e^2 \quad (\text{simplify using inverse function rules, } e^{\ln A} = A) \\
x &= \pm\sqrt{e^2} \\
x &= \pm e
\end{aligned}$$

Incorrect (incomplete solution)

$$\begin{aligned} \ln x^2 &= 2 \\ 2\ln x &= 2 \\ \ln x &= 1 \\ e^{\ln x} &= e^1 \\ x &= e \end{aligned}$$

We missed the $x = -e$ solution! This happened since the domain changed when we wrote $\ln x^2 = 2\ln x$.

Domain of $\ln x^2$ is $x^2 > 0$ which means $x \in (-\infty, 0) \cup (0, \infty)$.
Domain of $2\ln x$ is $x > 0$ which means $x \in (0, \infty)$.

More properly, we should have written $\ln x^2 = 2\ln|x|$ since domain of $2\ln|x|$ is $x \in (-\infty, 0) \cup (0, \infty)$.

$$\begin{aligned} \ln x^2 &= 2 \\ 2\ln|x| &= 2 \\ \ln|x| &= 1 \\ e^{\ln|x|} &= e^1 \\ |x| &= e \\ x &= \pm e \end{aligned}$$

Example Solve the equation $\dfrac{44}{1+4e^{-7x}} = 32$ algebraically.

Start by isolating the e^{-7x}:

$$\begin{aligned} \frac{44}{1+4e^{-7x}} &= 32 \\ \frac{1+4e^{-7x}}{44} &= \frac{1}{32} \\ 1+4e^{-7x} &= \frac{44}{32} \\ 1+4e^{-7x} &= \frac{11}{8} \\ 4e^{-7x} &= \frac{11}{8} - 1 \\ 4e^{-7x} &= \frac{3}{8} \end{aligned}$$

Now take logarithms of both sides:

$$e^{-7x} = \frac{3}{32}$$

Use $\ln e^A = A$:

$$\begin{aligned} \ln e^{-7x} &= \ln\left(\frac{3}{32}\right) \\ -7x &= \ln\left(\frac{3}{32}\right) \end{aligned}$$

You can have different forms of solution:

$$\begin{aligned} x &= -\frac{1}{7}\ln\left(\frac{3}{32}\right) \\ &= -\frac{1}{7}\ln\left((\frac{32}{3})^{-1}\right) \end{aligned}$$

This is in the domain of the original logistic equation, so it is a solution.

$$= \frac{1}{7}\ln\left(\frac{32}{3}\right)$$

Example Solve the equation $\ln(x-2) - \ln x = 3$ algebraically.

Start by writing as a single logarithm, use $\ln A - \ln B = \ln(A/B)$: $\quad \ln(x-2) - \ln x = 3$

$$\ln\left(\frac{x-2}{x}\right) = 3$$

Now take exponential of both sides: $\quad e^{\ln\left(\frac{x-2}{x}\right)} = e^3$

Now solve for x: $\quad \dfrac{x-2}{x} = e^3$

$$x - 2 = e^3 x$$

$$x - e^3 x = 2$$

$$x(1-e^3) = 2$$

$$x = \frac{2}{1-e^3}$$

Since $e > 2$, this number is actually less than zero. But in our original equation, we had to have $x > 0$ and $x - 2 > 0$ ($x > 2$) for the logarithms to be defined. These are both satisfied if $x > 2$. So, sadly, this equation has no solution, since the only solution we found was not greater than 2.

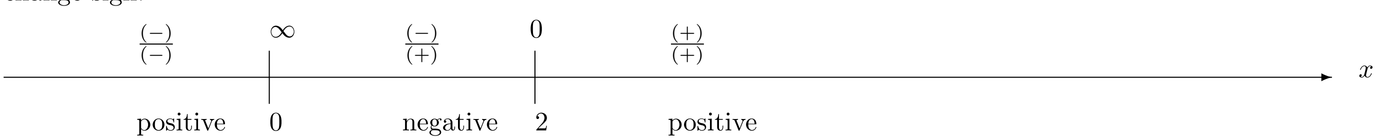
Example Solve the equation $\ln\left(\dfrac{x-2}{x}\right) = 3$ algebraically.

$$\begin{aligned}
\ln\left(\frac{x-2}{x}\right) &= 3 \\
e^{\ln\left(\frac{x-2}{x}\right)} &= e^3 \\
\frac{x-2}{x} &= e^3 \\
x-2 &= e^3 x \\
x - e^3 x &= 2 \\
x(1-e^3) &= 2 \\
x &= \frac{2}{1-e^3} \sim -0.104791 \text{ or, as above, we can figure out this is less than 0.}
\end{aligned}$$

Now, in our original equation, we require $(x-2)/x > 0$ for the logarithm to be defined.

This is an inequality! We can solve it using a sign chart.

The numerator is zero if $x = 2$, the denominator is zero if $x = 0$. These are the possible values where the function will change sign.



From the sign chart, we see that the inequality is satisfied if $x \in (-\infty, 0) \cup (2, \infty)$.

This is the domain of the function $\ln\left(\dfrac{x-2}{x}\right)$.

Since our solution $x = \dfrac{2}{1-e^3}$ is in the domain, it is not extraneous, it is a solution to the original problem.

Example Solve $\log_3 x + \log_3(x-19) = \log_3(20x)$.

First, notice the bases are all the same–if they weren't, this would be mighty difficult to solve!

$$\begin{aligned}
\text{Isolate a single logarithm:} \qquad \log_3 x + \log_3(x-19) &= \log_3(20x) \\
\log_3(x(x-19)) &= \log_3(20x) \\
\log_3(x(x-19)) - \log_3(20x) &= 0 \\
\log_3\left(\frac{x(x-19)}{20x}\right) &= 0 \\
3^{\log_3\left(\frac{x-19}{20}\right)} &= 3^0 \\
\text{Use } 3^{\log_3 A} = A: \qquad \frac{x-19}{20} &= 1 \\
x &= 39
\end{aligned}$$

Since $x = 39$ is in the domain of the functions in the original equation, this is a solution.

Example Solve $e^{2x} - e^x - 1 = 0$.

This is a tricky one to solve, since it requires us to recognize that it is actually a quadratic in e^x!

Here's the solution: Let $z = e^x$. Then $z^2 = e^{2x}$. Our equation becomes

$$\begin{aligned}
&z^2 - z - 1 = 0 \\
&\qquad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&\qquad\;\; = \frac{1 \pm \sqrt{1+4}}{2} \\
&\qquad\;\; = \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

Now, we can get x: Since $z = e^x$, we have $x = \ln z$.

$$x = \ln\left(\frac{1+\sqrt{5}}{2}\right) \qquad x = \ln\left(\frac{1-\sqrt{5}}{2}\right)$$

The solution with $x = \ln\left(\frac{1-\sqrt{5}}{2}\right)$ is extraneous, since this is not a real number (the number is negative, which is outside the domain of the logarithm).

The only solution is $x = \ln\left(\frac{1+\sqrt{5}}{2}\right)$.

Example How long does it take for 12g of carbon-14 in a tree trunk to be reduced to 10g of carbon-14 by radioactive decay? The half-life for carbon-14 is 5730 years.

The model for radioactive decay is $A(t) = A_0 e^{rt}$.

First, use the half-life to determine r:

$$\begin{aligned}
A(t) &= A_0 e^{rt} \\
\frac{1}{2} A_0 &= A_0 e^{r5730} \\
\frac{1}{2} &= e^{r5730} \\
\ln(\frac{1}{2}) &= \ln(e^{r5730}) \\
\ln(\frac{1}{2}) &= r5730 \\
r &= \frac{\ln(\frac{1}{2})}{5730} = -0.000120968
\end{aligned}$$

The model becomes $A(t) = A_0 e^{-0.000120968t}$. Now put in the value for A_0 and $A(t)$ and solve for t:

$$\begin{aligned}
10 &= 12e^{-0.000120968t} \\
\ln(10/12) &= \ln(e^{-0.000120968t}) \\
\ln(10/12) &= -0.000120968t \\
t &= \frac{\ln(10/12)}{-0.000120968} = 1507.19
\end{aligned}$$

There will be 10g left after 1507 years.

Example The level of a prescription drug built up in the human body over time can be found from the formula

$$L = \frac{D}{1-(0.5)^{n/h}}$$

where D is the amount taken every n hours and h is the drugs half-life in hours.

If a doctor wants the level of the drug lorazepam (half-life 14 hours) to build up to a level of 5.58 milligrams in a patient taking 2.5 milligram doses, then how often should the doses be taken?

$$\begin{aligned}
L &= \frac{D}{1-(0.5)^{n/h}} \\
5.58 &= \frac{2.5}{1-(0.5)^{n/14}} \\
1-(0.5)^{n/14} &= \frac{2.5}{5.58} \\
(0.5)^{n/14} &= 1 - \frac{2.5}{5.58} = 0.551971 \\
(0.5)^{n/14} &= 0.551971 \\
\ln[(0.5)^{n/14}] &= \ln[0.551971] \\
\frac{n}{14}\ln[0.5] &= -0.594259 \\
n &= \frac{14(-0.594259)}{\ln[0.5]} = 12.0027
\end{aligned}$$

The patient should take the dosage every 12 hours.