

**Concepts:** Graphical Solution, Algebraic: Substitution Method, Algebraic: Elimination Method.

A solution of a system of two equations in two variables is an ordered pair of real numbers that is a solution of each equation. Nonlinear systems can have multiple ordered pair solutions.

A solution of a system of two inequalities in two variables is a region in the  $xy$ -plane.

There are graphical and algebraic methods of solving systems of equations.

The textbook discusses the techniques for systems of equations in three sections, but the techniques are essentially the same in each section. I am including systems of inequalities here as well.

• **8.1 Systems of Linear Equations in Two Variables**

- graphical solution
- algebraic solution
  - \* substitution method
  - \* elimination method (addition method)

• **8.2 Systems of Linear Equations in Three Variables**

- no graphical solution for us (typically hard to sketch in  $\mathbb{R}^3$ )
- algebraic solution
  - \* substitution method
  - \* elimination method (addition method)

• **8.3 Nonlinear Systems of Equations**

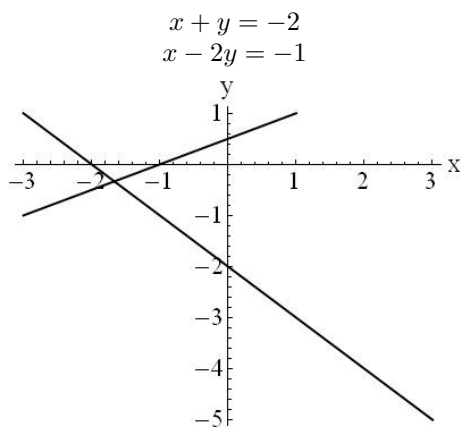
- graphical solution
- algebraic solution
  - \* substitution method

• **8.5 Inequalities and Systems of Inequalities in Two Variables**

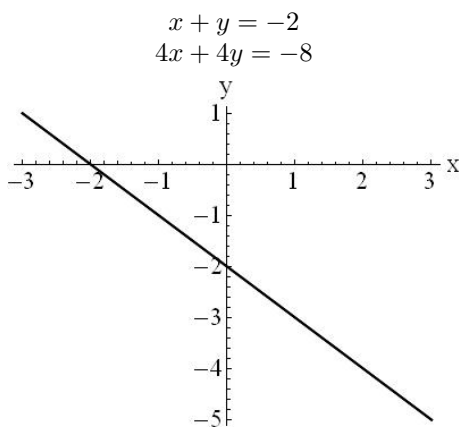
- graphical solution using test points
- no algebraic solution for us (Calculus III: if you need it, from sketch write the description of the region)

**Linear Equations**

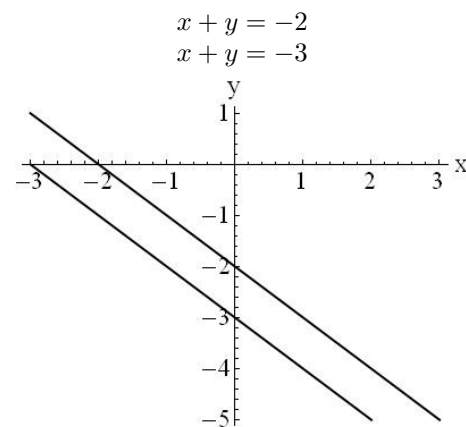
Linear equations in any dimension are kind of special. If you are solving linear equations in two dimensions, then graphically you have two lines in the  $xy$ -plane, and the solution is where they intersect. The solution can be either a single point, an infinite number of solutions, or no solution. We can see why graphically.



Two intersecting lines, one solution  
(independent)



The same line, infinite number of solutions  
(dependent)



Parallel lines, no solution  
(inconsistent)

### Graphical Solution

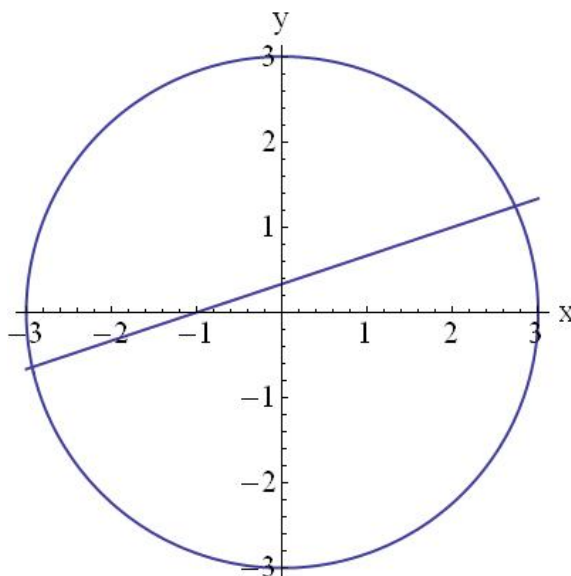
Graphically, the two equations are two curves in the  $xy$ -plane, and the solution to the system will be where the two curves intersect.

**Example** Solve the system of equations graphically.

$$\begin{aligned}x^2 + y^2 &= 9 \\x - 3y &= -1\end{aligned}$$

We need to graph the two equations. The first equation is a circle of radius 3 centered at  $(0, 0)$ .

Let's rewrite the second equation as  $y = \frac{1}{3}(x + 1) = \frac{x}{3} + \frac{1}{3}$ . This is a straight line, with slope  $1/3$  and  $y$ -intercept  $1/3$ .



We can see that there are two solutions, but we can only estimate what they are.

It looks like the solution to the system of equations is two points, which are approximately  $(2.8, 1.2)$  and  $(-2.9, -0.7)$ . If we graphed these using a calculator and zoomed in on the points of intersection, we could get more accuracy.

Graphical solutions are good to verify a solution you have found, but cannot produce an exact solution in general. We need algebraic techniques.

### The Substitution Method

This method involves the following steps:

1. solve one of the equations for one of the unknown variables,
2. substitute the equation from step (1) into the other equation to produce a single equation in a single unknown variable,
3. solve this equation for the unknown variable,
4. substitute into the equation from step (1) to get the second unknown variable.

This procedure may not work if it is difficult to perform step (3).

**Example** Solve the system of equations using the method of substitution.

$$\begin{aligned}x^2 + y^2 &= 9 \\x - 3y &= -1\end{aligned}$$

We have a choice to make, which equation to begin with. If there is a linear equation, begin with that one.

$$\begin{aligned}x - 3y &= -1 \\x &= -1 + 3y\end{aligned}$$

Now, substitute this into the other equation, and solve for  $y$ :

$$\begin{aligned}x^2 + y^2 &= 9 \\(-1 + 3y)^2 + y^2 &= 9 \\1 + 9y^2 - 6y + y^2 &= 9 \\10y^2 - 6y - 8 &= 0 \\5y^2 - 3y - 4 &= 0 \\y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{3 \pm \sqrt{(-3)^2 - 4(5)(-4)}}{2(5)} \\&= \frac{3 \pm \sqrt{9 + 80}}{10} \\&= \frac{3 \pm \sqrt{89}}{10}\end{aligned}$$

Now, we substitute the two solutions we found for  $y$  to determine the corresponding  $x$ :

$$\begin{aligned}x &= -1 + 3y \\&= -1 + 3\left(\frac{3 + \sqrt{89}}{10}\right) \\&= \left(\frac{-10 + 9 + 3\sqrt{89}}{10}\right) = \left(\frac{-1 + 3\sqrt{89}}{10}\right)\end{aligned}$$

So a solution to the system is  $\left(\frac{-1 + 3\sqrt{89}}{10}, \frac{3 + \sqrt{89}}{10}\right)$ .

Get the other solution:

$$\begin{aligned}x &= -1 + 3y \\&= -1 + 3\left(\frac{3 - \sqrt{89}}{10}\right) \\&= \left(\frac{-10 + 9 - 3\sqrt{89}}{10}\right) \\&= \left(\frac{-1 - 3\sqrt{89}}{10}\right)\end{aligned}$$

So another solution to the system is  $\left(\frac{-1 - 3\sqrt{89}}{10}, \frac{3 - \sqrt{89}}{10}\right)$ .

Compare with our graphical estimation:

$$\left(\frac{-1 + 3\sqrt{89}}{10}, \frac{3 + \sqrt{89}}{10}\right) \sim (2.73, 1.24).$$
$$\left(\frac{-1 - 3\sqrt{89}}{10}, \frac{3 - \sqrt{89}}{10}\right) \sim (-2.93, -0.64).$$

### The Elimination Method

This method works best on linear systems, although it can work in other instances.

The process:

1. rewrite one of the equations so that a coefficient of one of the variables is the opposite (different sign) from the other equation,
2. add the two equations, which will eliminate one of the variables,
3. solve the resulting equation for the unknown variable,
4. use one of the original equations to solve for the other unknown variable.

**Example** Solve the system of equations using elimination:

$$\begin{aligned}2x + y &= 10 \\x - 2y &= -5\end{aligned}$$

Rewrite the first equation by multiplying by 2, then add the equations

$$\begin{aligned}4x + 2y &= 20 \\x - 2y &= -5 \\5x &= 15\end{aligned}$$

Solve the new equation for  $x$ :  $x = 3$ .

Use the first original equation to determine  $y$ :

$$\begin{aligned}2x + y &= 10 \\2(3) + y &= 10 \\y &= 10 - 6 = 4\end{aligned}$$

The solution to the system is  $(3, 4)$ , or  $x = 3, y = 4$ .

### Solving System of Inequalities Graphically

In this section we will look at solving inequalities by graphing and identifying the region that satisfies all the inequalities. In calculus, this skill will be useful if you need to determine the region enclosed by  $y = f(x)$  and  $y = g(x)$ , which is a first step when performing volumes of rotation (in Calculus II).

If we stick to linear inequalities, we are led not to a calculus application but instead to the topic *Linear Programming*, where we are interested in optimizing (finding max or min) a given function subject to a variety of constraints (the inequalities). In this situation, the region that satisfies all the constraints is called the *feasible region*. For Precalculus, we will not do any linear programming, although linear programming is one of the most important mathematical applications ever developed.

### Steps to Solving System of Inequalities Graphically

Consider the system of inequalities (note we could have  $\leq, \geq, <, >$  in each case):

$$F(x, y) < 0$$

$$G(x, y) > 0$$

For  $F(x, y) < 0$ :

1. Sketch the equality  $F(x, y) = 0$ . Use dashed line if you have  $<$  or  $>$  and solid line if you have  $\leq, \geq$ . The dashed line indicates that the boundary is not part of the set of points that satisfies the inequality.
2. Pick a test point not on the line, and see if the inequality is true at that point. If it is, shade the side of the curve that contains the test point. If it is false, shade the side of the curve that does not contain the test point.

Repeat the procedure for the other inequality.

Identify the region that satisfies all the inequalities.

If asked, determine any points of intersection where the boundary of the region changes from one curve to another. This might be very difficult or even impossible to do! This final step is one you would need to do in Calculus III, where you need to be able to describe regions more precisely than just in a sketch. For Precalculus, getting the sketch is sufficient.

**Example** Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region.

$$x - 3y - 6 \leq 0$$

$$y + x^2 + 2x \geq 2$$

Sketch  $x - 3y - 6 = 0$

If  $x=0$ , then  $-3y - 6 = 0$   
 $y = -2$

$\Rightarrow (0, -2)$

If  $y=0$ , then  $x - 6 = 0$   
 $x = 6$

$\Rightarrow (6, 0)$

Test point  $(0, 0)$ :

$$(0) - 3(0) - 6 \leq 0$$

$$-6 \leq 0 \text{ True!}$$

shade the side of line containing  $(0, 0)$ .

Sketch  $y + x^2 + 2x = 2$   
 quadratic. Complete square.

$$\begin{aligned} y &= -x^2 - 2x + 2 \\ &= -(x^2 + 2x - 2) \\ &= -\left(\underbrace{x^2 + 2x + 1}_{(x+1)^2} - 1 - 2\right) \\ &= -\left([x+1]^2 - 3\right) \\ &= -[x+1]^2 + 3 \end{aligned}$$

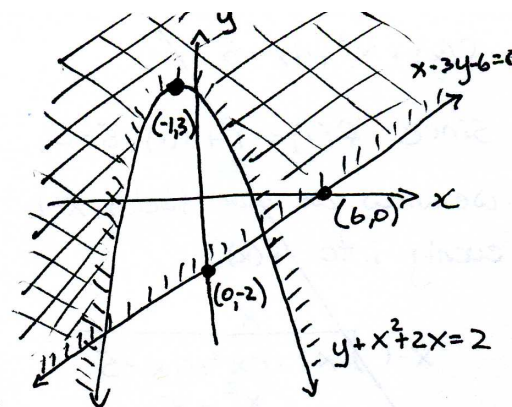
opens down, vertex  $(-1, 3)$

Test point  $(0, 0)$ :

$$(0) + (0)^2 + 2(0) \geq 2$$

$$0 \geq 2 \text{ False!}$$

shade side of curve not containing  $(0, 0)$ .



The region is the cross hatched area. Notice getting the sketch of the region relies on sketching techniques and using a test point to determine which side of a curve satisfies an inequality.

To determine the points of intersection, solve

$$x - 3y - 6 = 0,$$

$$y + x^2 + 2x = 2.$$

Take the first equation and solve for  $x$ , substitute into the second equation. Then solve for  $y$ :

$$x = 3y + 6 \text{ (1st equation solved for } x\text{)}$$

$$y + x^2 + 2x = 2 \text{ (2nd equation)}$$

$$y + (3y + 6)^2 + 2(3y + 6) = 2 \text{ (2nd equation with } x \text{ replaced)}$$

$$y + 9y^2 + 36y + 36 + 6y + 12 - 2 = 0$$

$$9y^2 + 43y + 46 = 0$$

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-43 \pm \sqrt{(-43)^2 - 4(9)(46)}}{2(9)} \\ y &= \frac{-43 \pm \sqrt{193}}{18} \end{aligned}$$

Substitute these values of  $y$  into the first equation to get the ordered pairs:

If  $y = \frac{-43 + \sqrt{193}}{18}$ , then  $x = 3y + 6 = 3\left(\frac{-43 + \sqrt{193}}{18}\right) + 6 = \frac{-7 + \sqrt{193}}{6}$ . So one solution is

$$\left( \frac{-7 + \sqrt{193}}{6}, \frac{-43 + \sqrt{193}}{18} \right).$$

If  $y = \frac{-43 - \sqrt{193}}{18}$ , then  $x = 3y + 6 = 3\left(\frac{-43 - \sqrt{193}}{18}\right) + 6 = \frac{-7 - \sqrt{193}}{6}$ . So another solution is

$$\left( \frac{-7 - \sqrt{193}}{6}, \frac{-43 - \sqrt{193}}{18} \right).$$

Our solution is complete.

If needed, we could now describe the region in words as follows: The region is above the curve  $y + x^2 + 2x = 2$  for  $x \in [1/6(-7 - \sqrt{193}), 1/6(-7 + \sqrt{193})]$ , and above the curve  $x - 3y - 6 = 0$  for all other  $x$ .

Aside: *Mathematica* (those of you taking Calculus I will learn how to use *Mathematica*) can create plots of inequalities using the command **RegionPlot**.