

Arithmetic of Real Numbers

- Division of a real number other than zero by 0 is undefined

$$\frac{456}{0} = \text{undefined}$$

- Exponents—pay attention to the base!

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16$$

$$-2^4 = -(2)(2)(2)(2) = -16$$

- Order of Operations: practice order of operations until it becomes second nature to you.

1. Do all operations inside parentheses.
2. Raise numbers to a power.
3. Multiply and Divide from left to right.
4. Add and subtract from left to right.

Arithmetic of Algebraic Expressions

- An algebraic expression contains numbers and variables.

Distributive property (notice that once you distribute, the parentheses are not needed):

$$x(b + c) = xb + xc.$$

- Factors are quantities that are multiplied together:

xyz has factors x, y , and z .

$x(x + 6)$ has factors x and $x + 6$.

- Terms are quantities that are added together:

$a + b + c + d$ has terms a, b, c and d .

$17x^3 + 8b + \frac{1}{c} + (-14)$ has terms $17x^3, 8b, \frac{1}{c}$ and -14 .

Sometimes you may see

$x - 14$ has terms x and 14 ,

which is the same as

$x + (-14)$ has terms x and -14 .

I prefer the latter since you will be less likely to make sign errors if you keep track of the sign by including it in the term. You can also ensure you keep a minus sign with the factor by writing

$$-18(x - 1) = (-18)(x - 1) = -18x + 18.$$

This will help you avoid the mistake that leads you to $-18x - 18$.

- Like terms are terms with identical variables and exponents, but the numbers can be different. You must practice identifying like terms.

$16x^3, -\frac{1}{5}x^3$ and x^3 are all like terms.

$x^2y^4, -24x^2y^4, \frac{1}{2}x^2y^4$ are all like terms.

When substituting into variable expressions, it can help to use brackets but not fill in what goes into them right away. This can help you avoid some sign errors that might occur if you aren't careful with your substitution.

Example Evaluate $\frac{x-1}{1+x^2}$ when $x = -1$.

$$\begin{aligned} \frac{x-1}{1+x^2} &= \frac{(\quad)-1}{1+(\quad)^2} \text{ (put parenthesis where there were } x\text{'s)} \\ &= \frac{(-1)-1}{1+(-1)^2} \text{ (put the value for } x \text{ in the parentheses)} \\ &= \frac{-1-1}{1+(-1)(-1)} \text{ (simplify)} \\ &= \frac{-2}{1+1} = \frac{-2}{2} = -1 \end{aligned}$$

Example Density d , mass m and volume V are related by the formula $d = \frac{m}{V}$. When working with formulas where the variables have physical meaning, you should include units in your calculations.

What is the density of mercury if you measure that 24 milliliters of mercury has a mass of 326 grams?

$$\begin{aligned} d &= \frac{m}{v} \text{ write down equation you will use} \\ d &= \frac{(326 \text{ grams})}{(24 \text{ milliliters})} \text{ substitute values, including units} \\ d &= \frac{163 \times \cancel{2} \text{ grams}}{12 \times \cancel{2} \text{ milliliters}} \text{ simplify} \\ d &= \frac{163}{12} \frac{\text{grams}}{\text{milliliters}} \text{ convert to decimal at the last step} \\ d &= 13.5833 \frac{\text{grams}}{\text{milliliters}} \end{aligned}$$

If you performed this measurement, you would also need to do an error analysis. You could convert to a decimal earlier if you do the error analysis. Rounding and error analysis is something you will study in your science classes.

Temperature conversion

$$T_C = \frac{5}{9}(T_F - 32)$$

where T_C is temperature in Celsius, T_F is temperature in Fahrenheit.

What is $-40F$ in Celsius?

$$\begin{aligned} T_C &= \frac{5}{9}(T_F - 32) \\ T_C &= \frac{5}{9}(-40 - 32) \\ T_C &= \frac{5}{9}(-72) \\ T_C &= -\frac{5 \times 8 \times \cancel{9}}{\cancel{9}} \\ T_C &= -40 \end{aligned}$$

So $-40F = -40C$. Note: We typically don't include units in this calculation since the $\frac{5}{9}$ really has units $\frac{5C}{9F}$ and it is a simple calculation so we just leave them out.

Removing grouping symbols by using the distributive property and collecting like terms.

$$\begin{aligned} 5(4x - 3(x - 2)) &= 5(4x - 3x + 6) \\ &= 5(x + 6) \\ &= 5x + 30 \end{aligned}$$

When removing parentheses, many text book say quite emphatically that you should “Remember to remove the innermost parentheses first”. This is simply advice, not a mathematical rule like the order of operations. The order of operations cannot be changed, but with parentheses you could work from outermost parentheses first if you use the distributive property correctly.

Wikipedia has this to say about removing parentheses, and I think is a good way to view algebraic simplification:

*If an expression involves parentheses, then do the arithmetic inside the innermost pair of parentheses first and work outward, or use the distributive law to remove parentheses.*¹

In my opinion, the following is one of the most important concepts in mathematics, and one that we sometimes fail to teach to students:

Your goal should be to understand what are the rules of mathematics that you cannot break, and what is simply advice to make problem solving go faster. There are many different paths to solution, and any path that uses mathematics correctly is a good one.

Comments

• Never write two “operations” without using bracket! What I mean is, never write $34 + -56$, since it obscures the fact that you are adding negative fifty-six. Other examples of poor notation include:

$$5 \times -7 \text{ should be written as } 5 \times (-7) = -5(7).$$

$$5 \div -7 \text{ should be written as } 5 \div (-7) = -\frac{5}{7}.$$

$$5 - -7 \text{ should be written as } 5 - (-7) = 5 + 7.$$

• Here is how we read the mathematics:

$34 - 56$ is read as “thirty-four minus fifty-six”.

$34 + (-56)$ is read as “thirty four plus negative fifty-six”.

In this way, we can always think of subtraction as adding a negative number. This may seem like a very subtle point, but it will pay big rewards later when we start dealing with variables.

¹http://en.wikipedia.org/wiki/Order_of_operations retrieved on Aug 23, 2007.

Examples

Example Simplify $\left(\frac{1}{2}\right)^3 + \frac{1}{4} - \left(\frac{1}{6} - \frac{1}{12}\right) - \frac{2}{3} \cdot \left(\frac{1}{4}\right)^2$.

This is all about getting the order of operation right.

$$\begin{aligned} \left(\frac{1}{2}\right)^3 + \frac{1}{4} - \left(\frac{1}{6} - \frac{1}{12}\right) - \frac{2}{3} \cdot \left(\frac{1}{4}\right)^2 &= \left(\frac{1}{2}\right)^3 + \frac{1}{4} - \left(\frac{2}{12} - \frac{1}{12}\right) - \frac{2}{3} \cdot \left(\frac{1}{4}\right)^2 \quad (\text{do parentheses}) \\ &= \left(\frac{1}{2}\right)^3 + \frac{1}{4} - \frac{1}{12} - \frac{2}{3} \cdot \left(\frac{1}{4}\right)^2 \quad (\text{do powers}) \\ &= \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) + \frac{1}{4} - \frac{1}{12} - \frac{2}{3} \cdot \left(\frac{1}{4}\right) \left(\frac{1}{4}\right) \\ &= \frac{1}{8} + \frac{1}{4} - \frac{1}{12} - \frac{2}{3} \cdot \left(\frac{1}{16}\right) \quad (\text{do multiplication from left to right}) \\ &= \frac{1}{8} + \frac{1}{4} - \frac{1}{12} - \frac{1}{24} \quad (\text{do addition/subtraction from left to right}) \end{aligned}$$

LCD is 24

$$\begin{aligned} &= \frac{3}{24} + \frac{6}{24} - \frac{2}{24} - \frac{1}{24} \\ &= \frac{3 + 6 - 2 - 1}{24} \\ &= \frac{6}{24} = \frac{1}{4} \quad (\text{simplify}) \end{aligned}$$

Example Simplify $(-2)^3 - 4 \left| \frac{1}{4} - \frac{1}{3} \right| - 6 \cdot \left(\frac{1}{4}\right)^2$.

$$\begin{aligned} (-2)^3 - 4 \left| \frac{1}{4} - \frac{1}{3} \right| - 6 \cdot \left(\frac{1}{4}\right)^2 &= (-2)^3 - 4 \left| \frac{3}{12} - \frac{4}{12} \right| - 6 \cdot \left(\frac{1}{4}\right)^2 \quad (\text{do parentheses}) \\ &= (-2)^3 - 4 \left| -\frac{1}{12} \right| - 6 \cdot \left(\frac{1}{4}\right)^2 \quad (\text{do absolute value (it's like a parentheses)}) \\ &= (-2)^3 - 4 \left(\frac{1}{12}\right) - 6 \cdot \left(\frac{1}{4}\right)^2 \quad (\text{do powers}) \\ &= (-2)(-2)(-2) - 4 \left(\frac{1}{12}\right) - 6 \cdot \left(\frac{1}{4}\right) \left(\frac{1}{4}\right) \\ &= -8 - 4 \left(\frac{1}{12}\right) - 6 \cdot \left(\frac{1}{16}\right) \quad (\text{do multiplication from left to right}) \\ &= -8 - \frac{1}{3} - \frac{3}{8} \quad (\text{do addition/subtraction from left to right}) \end{aligned}$$

LCD is 24

$$\begin{aligned} &= -\frac{192}{24} - \frac{8}{24} - \frac{9}{24} \\ &= \frac{-192 - 8 - 9}{24} \\ &= \frac{-209}{24} \end{aligned}$$

NOTE: Because I am looking for an improper fraction as my answer, I never really do any division. This is why I say “do multiplication from left to right” rather than “do multiplication/division from left to right”. If I want a decimal answer, I can do a division at the end to get it. This is typical of what you will do, since is more common to see quantities like $\frac{1}{4}$ rather than $1 \div 4$.

Compare the following two problems, both which represent the same number:

$$\begin{aligned}(3 - 7)^2 \div 8 + 3 &= (-4)^2 \div 8 + 3 \text{ (do the parentheses)} \\ &= (-4)(-4) \div 8 + 3 \text{ (do the powers)} \\ &= 16 \div 8 + 3 \text{ (do the multiplication/division from left to right)} \\ &= 2 + 3 \text{ (do the addition/subtraction from left to right)} \\ &= 5 \text{ (simplify)}\end{aligned}$$

$$\begin{aligned}\frac{(3 - 7)^2}{8} + 3 &= \frac{(-4)^2}{8} + 3 \text{ (do the parentheses)} \\ &= \frac{(-4)(-4)}{8} + 3 \text{ (do the powers)} \\ &= \frac{16}{8} + 3\end{aligned}$$

LCD is 8

$$\begin{aligned}&= \frac{16}{8} + \frac{24}{8} \text{ (do the the addition/subtraction from left to right)} \\ &= \frac{16 + 24}{8} \text{ (do the the addition/subtraction from left to right)} \\ &= \frac{40}{8} \text{ (simplify)} \\ &= 5\end{aligned}$$

Both techniques are correct. I am showing a lot of steps for those students who need it, if you can do some of the arithmetic in you head *correctly*, then you don't need to show nearly this much work.

Example A square part is needed for a newly designed car, but the size of the square is not known yet. What is known is that it must be $3x$ cm on each side, where x is a variable that whose numerical value will be determined later by the engineers.

To save space, the engineers are considering shortening the square part by 1cm on one side, making it a rectangle. By how much will this reduce the area of the part? Verify your answer makes sense when $x = 4$ cm.

First, we need to determine the area of the original part, which is $(3x)^2 = (3x)(3x) = 9x^2$ cm².

If they shrink the square by 1cm on one side to make it a rectangle, the area will be $(3x)(3x - 1) = 9x^2 - 3x$ cm².

The area of the part is will have been reduced by $3x$ cm².

Example Simplify $2a - (6b - 4(a - (b - 3a)))$.

$$\begin{aligned}2a - [6b - 4(a - (b - 3a))] &= 2a - [6b + (-4)(a - (b - 3a))] \\ &\quad \text{(use parentheses with negative factor } -4, \text{ just to ensure we get the sign right)} \\ &= 2a - [6b + (-4)(a) - (-4)(b - 3a)] \\ &= 2a - [6b - 4a + 4(b - 3a)] \\ &= 2a - [6b - 4a + 4b - 12a] \text{ (distribute the } -1) \\ &= 2a - 6b + 4a - 4b + 12a \\ &= 2a + 4a + 12a - 6b - 4b \\ &= 18a - 10b\end{aligned}$$

What if we didn't work from innermost parentheses to outermost as the text suggests?

$$\begin{aligned}
 2a - [6b - 4(a - (b - 3a))] &= 2a - 6b + 4(a - (b - 3a)) \\
 &= 2a - 6b + 4a - 4(b - 3a) \\
 &= 2a - 6b + 4a - 4b + 12a \\
 &= 2a + 4a + 12a - 6b - 4b \\
 &= 18a - 10b
 \end{aligned}$$

Working from innermost parentheses first is usually what to do since it makes the simplifying easier. In this case it really didn't.

Example Grandpa and Grandma have a tradition of eating out once a week. The average cost of the meal was \$20. In Massachusetts there is a 5% sales tax added on to the cost of the meal. The couple always left a 15% tip. They continued this pattern for 10 years. Grandma felt they should calculate the tip based on the total of the cost of the meal including the sales tax, but Grandpa felt the tip should be calculated on the cost of the meal alone. How much difference would this make over the ten year period?

For each meal, here is the amount of the tip (on average):

$$\begin{aligned}
 \text{avg cost of meal} &= \$20.00 \\
 \text{amount of sales tax} &= \text{avg cost of meal} \times 5\% \\
 &= \$20.00 \times 0.05 \\
 &= \$1.00 \\
 \text{amount of tip} &= (\text{avg cost of meal} + \text{amount of sales tax}) \times 15\% \\
 &= (\$20.00 + \$1.00) \times 0.15 \\
 &= (\$21.00)(0.15) \\
 &= \$3.15
 \end{aligned}$$

$$\begin{aligned}
 \text{Amount tipped over 10 years} &= 52 \text{ weeks in a year} \times 10 \text{ years} \times \text{tip per week} \\
 &= 52 \times 10 \times \$3.15 \\
 &= \$1638.00
 \end{aligned}$$

If they did not include the sales tax in their tip, this is how much they would have tipped:

$$\begin{aligned}
 \text{avg cost of meal} &= \$20.00 \\
 \text{amount of tip} &= (\text{avg cost of meal}) \times 15\% \\
 &= (\$20.00) \times 0.15 \\
 &= \$3.00
 \end{aligned}$$

$$\begin{aligned}
 \text{Amount tipped over 10 years} &= 52 \text{ weeks in a year} \times 10 \text{ years} \times \text{tip per week} \\
 &= 52 \times 10 \times \$3.00 \\
 &= \$1560.00
 \end{aligned}$$

They would have saved $\$1638 - \$1560 = \$78$.

Example Simplify $3\{3b^2 + 2[5b - (2 - b)]\}$.

Note: when you have lots of parentheses, use size to keep it clear where the sets of parentheses are. I am going to use the distributive law from outermost parentheses first to simplify.

$$\begin{aligned} 3\{3b^2 + 2[5b - (2 - b)]\} &= 3(3b^2) + 3(2)[5b - (2 - b)] \text{ (distribute the 3)} \\ &= 9b^2 + 6[5b - (2 - b)] \text{ (simplify)} \\ &= 9b^2 + 6(5b) - 6(2 - b) \text{ (distribute the 6)} \\ &= 9b^2 + 30b + (-6)(2 + (-b)) \text{ (simplify)} \\ &= 9b^2 + 30b + (-6)2 + (-6)(-b) \text{ (distribute the -6)} \\ &= 9b^2 + 30b + (-12) + (+6b) \text{ (simplify)} \\ &= 9b^2 + 30b - 12 + 6b \text{ (simplify)} \\ &= 9b^2 + 36b - 12 \end{aligned}$$

You can show fewer steps, as long as you don't make any errors.