Exponents

• Rules of Exponents:

- $x^0 = 1$ if $x \neq 0$ (0^0 is indeterminant and is dealt with in calculus).
- Product Rule: $x^a \cdot x^b = x^{a+b}$.
- Quotient Rule: $\dfrac{x^a}{x^b} = x^{a-b}$.
- Power Rule: $(x^a)^b = x^{ab}$.
- Product Raised to Power Rule: $(xy)^a = x^a y^a$.
- Quotient Raised to a Power Rule: $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$ if $y \neq 0$.
- Negative Exponent: $x^{-n} = \dfrac{1}{x^n}$, if $x \neq 0$.

The rules of exponents are *tremendously important*, and it is critical that you memorize them. They will be useful in a variety of ways for the rest of the course and beyond.

Notice I did not break the quotient rule up into a bunch of cases–that isn't necessary if you are comfortable working with negative exponents (which you will be!).

Example Simplify by combining exponents in $\left(\dfrac{3x^{-3}y^{-2}}{x^{-2}y^{-4}}\right)^2$. Make all exponents positive in your final answer.

$$\left(\frac{3x^{-3}y^{-2}}{x^{-2}y^{-4}}\right)^2 = \frac{3^2(x^{-3})^2(y^{-2})^2}{(x^{-2})^2(y^{-4})^2} = \frac{9x^{-6}y^{-4}}{x^{-4}y^{-8}} = 9x^{-6+4}y^{-4+8} = 9x^{-2}y^4 = \frac{9y^4}{x^2}$$

Radical Notation and Rules of Radicals

If x is a nonnegative real number, then $\sqrt{x} > 0$ is the principal square root of x. This is because $(\sqrt{x})^2 = x$.

Higher order roots are defined using radical notation as: $\sqrt[n]{x}$.

In words, to evaluate the expression $\sqrt[n]{x} = y$ means you are looking for a number y that when multiplied by itself n times gives you the quantity x.

$\sqrt[4]{16} = 2$ since $(2)(2)(2)(2) = 16$.

Note that is is true that $(-2)(-2)(-2)(-2) = 16$ but we choose $+2$ since we want the principal root.

Rules of Radicals Working with radicals is important, but looking at the rules may be a bit confusing. Here are examples to help make the rules more concrete.

1. If x is a **positive real number**, then
   - $\sqrt[n]{x}$ is the nth root of x and $\left(\sqrt[n]{x}\right)^n = x$,

     $\left(\sqrt[3]{17}\right)^3 = 17$

     $\left(\sqrt[3]{8}\right)^3 = 8$ since $\sqrt[3]{8} = 2$ since $(2)(2)(2) = 8$
   - if n is a positive integer, we can write $x^{1/n} = \sqrt[n]{x}$.

     $8^{1/4} = \sqrt[4]{8}$

     $625^{1/4} = \sqrt[4]{625} = 5$ since $(5)(5)(5)(5) = 625$
2. If x is a **negative real number**, then

- $(\sqrt[n]{x})^n = x$ when n is an odd integer,

  $\left(\sqrt[3]{-6}\right)^3 = -6$

  $\left(\sqrt[3]{-8}\right)^3 = -8$ since $\sqrt[3]{-8} = -2$ since $(-2)(-2)(-2) = -8$
- $(\sqrt[n]{x})^n$ is not a real number when n is an even integer.

  $\left(\sqrt[2]{-6}\right)^2$ is not a real number (there is no real number that you can square and get a negative number)

3. For **all real numbers x (including negative values)**
   - $\sqrt[n]{x^n} = |x|$ when n is an even positive integer,

     $\sqrt[4]{(-16)^4} = |-16| = 16$ since because you take fourth power first, you are removing the negative sign

     $\sqrt[2]{(-6)^2} = \sqrt[2]{36} = \sqrt[2]{6^2} = 6$
   - $\sqrt[n]{x^n} = x$ when n is an odd positive integer.

     $\sqrt[3]{(-19)^3} = -19$

     $\sqrt[3]{(-8)^3} = \sqrt[3]{(-8)(-8)(-8)} = -8$

Product Rule for radicals: When a, b are nonnegative real numbers, $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$ (which is really just the exponent rule $(ab)^n = a^n b^n$).

Quotient Rule for radicals: When a, b are nonnegative real numbers (and $b \neq 0$), $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.

Example Evaluate $\left(\dfrac{16}{81}\right)^{3/4}$.

Solution: Since the radical for this expression would be $\left(\sqrt[4]{\dfrac{16}{81}}\right)^3$, we should look for a way to write 16/81 as (something)^4.

$$\begin{aligned}
\left(\frac{16}{81}\right)^{3/4} &= \left(\left(\frac{2}{3}\right)^4\right)^{3/4} \\
&= \left(\frac{2}{3}\right)^{4(3/4)} \\
&= \left(\frac{2}{3}\right)^3 \\
&= \frac{2^3}{3^3} = \frac{8}{27}
\end{aligned}$$

Notice that writing this as $\left(\dfrac{16}{81}\right)^{3/4} = \sqrt[4]{\left(\dfrac{16}{81}\right)^3} = \sqrt[4]{\dfrac{4096}{531441}}$ is mathematically true, it doesn't help us simplify.

Example Simplify $\sqrt{8} + \sqrt{50} - 2\sqrt{72}$.

Since we are dealing with square roots, we simplify by looking for quantities that can be written as (something)^2.

$$\begin{aligned}
\sqrt{8} + \sqrt{50} - 2\sqrt{72} &= \sqrt{4 \cdot 2} + \sqrt{25 \cdot 2} - 2\sqrt{36 \cdot 2} \\
&= \sqrt{2^2 \cdot 2} + \sqrt{5^2 \cdot 2} - 2\sqrt{6^2 \cdot 2} \\
&= \sqrt{2^2}\sqrt{2} + \sqrt{5^2}\sqrt{2} - 2\sqrt{6^2}\sqrt{2} \\
&= 2\sqrt{2} + 5\sqrt{2} - 2 \cdot 6\sqrt{2} \\
&= -5\sqrt{2}
\end{aligned}$$

Rationalizing

Rationalizing something means getting rid of any radicals.

• To rationalize a numerator, you want to modify the expression so as to remove any radicals from the numerator.

• To rationalize a denominator, you want to modify the expression so as to remove any radicals from the denominator.

• The expression $a + \sqrt{b}$ has the conjugate expression $a - \sqrt{b}$, which can be useful when rationalizing a denominator or numerator. For example, $2 - \sqrt{43 - x}$ and $2 + \sqrt{43 - x}$ are conjugate expressions.

Example Rationalize the denominator in the expression $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-2\sqrt{y}}$.

To rationalize the denominator, we multiply both the numerator and denominator by the conjugate of the denominator

$$\begin{aligned}
\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-2\sqrt{y}} &= \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-2\sqrt{y}} \cdot \frac{\sqrt{x}+2\sqrt{y}}{\sqrt{x}+2\sqrt{y}} \\
&= \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}+2\sqrt{y})}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}+2\sqrt{y})} \\
&= \frac{x+3\sqrt{x}\sqrt{y}+2y}{x-4y}
\end{aligned}$$

If we wanted to, we could also rationalize the numerator by multiplying both the numerator and denominator by the conjugate of the numerator.

$$\begin{aligned}
\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-2\sqrt{y}} &= \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-2\sqrt{y}} \cdot \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}} \\
&= \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}-\sqrt{y})} \\
&= \frac{x-y}{x-3\sqrt{x}\sqrt{y}+2y}
\end{aligned}$$

• *Simplified Form* for a radical expression is defined in the text, and includes the fact that there are no radicals in the denominator. I have a problem with this (this is a standard definition, not just in our text). The problem is, what do we mean by *simplified*? Sometimes we may want to get rid of radicals in the denominator, but sometimes we may want to get rid of radicals in the numerator. To say one is more simplified than the other is completely misguided, in my opinion.

So the real question is–what do we mean by simplified? Here's my answer:

> To simplify an expression may mean different things in different situations. I view something as simplified if it is in a form that makes the next thing you want to do with it easier. Do you want to sketch a function? Look for roots? Find a numeric value? Substitute it into something else? Have someone easily recognize the magnitude of a number? Depending on what you want to do, you may want slightly different forms. **Your goal is to become proficient with the algebraic techniques of simplification (rationalizing numerator, rationalizing denominator, finding a common denominator, factoring, etc), so you can easily do whatever simplification is required for the task at hand.**